From kinetic theory to fluid dynamics II

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The problem of plasma kinetic description is formally established

The Boltzmann equation provides the distribution function motion for each plasma species, for appropriate collision terms, as

$$\frac{\partial f_\alpha}{\partial t} + \nabla_{\boldsymbol{r}} \cdot (\boldsymbol{v} f_\alpha) + \nabla_{\boldsymbol{v}} \cdot (\frac{\boldsymbol{F}}{m_\alpha} f_\alpha) = C(f_\alpha)$$

$$f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t) \qquad \alpha = e, i, a$$

The(probability) distribution function provides the charge density ρ_C (r, t) and the electric current density J_C (r, t)

$$\rho_c(\boldsymbol{r},t) = \sum_\alpha \rho_{e\alpha} = \sum_\alpha q_\alpha n_\alpha(\boldsymbol{r},t)$$

$$\boldsymbol{J}_c(\boldsymbol{r},t) = \sum_\alpha \boldsymbol{J}_{e\alpha} = \sum_\alpha q_\alpha n_\alpha \boldsymbol{u}_\alpha = q_\alpha \int_{-\infty}^{+\infty} f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)\, \boldsymbol{v}\, d\boldsymbol{v}$$

That are introduced into the Maxwell equations *(ONLY TWO MOMENTS OF f)* giving rise to the coupling of plasma dynamics and macroscopic (out-Debye-sphere) fields :

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_c}{\varepsilon_o} \qquad \nabla \wedge \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \wedge \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{\boldsymbol{J}_c}{\varepsilon_o c^2}$$

Using both meaningful averages and Maxwell´s Eq. all plasma properties would be calculated through ***f***, this seems to be a **well posed problem**, with a **closed system of equations**, BUT…

It is extremely difficult to solve, even for C(f)= 0 !! (Vlasov Eq.)

Plasma physical description is set by means of the *f*-averages and macroscopic and thermal speeds

The particle velocity is expressed as the sum of a thermal speed w and the macroscopic fluid velocity $\boldsymbol{u}_\alpha$ (r, t) –first-order moment of *f* - as,

$$\boldsymbol{v} = \boldsymbol{u}_\alpha(\boldsymbol{r},t) + \boldsymbol{w} \qquad \alpha = e, i, a \qquad \langle \boldsymbol{v} \rangle = \boldsymbol{u}_\alpha(\boldsymbol{r},t) + \langle \boldsymbol{w} \rangle$$

with, $< w_x >=< w_y >=< w_z >= 0$ and, $\langle \boldsymbol{w} \bullet \boldsymbol{w} \rangle = \langle w^2 \rangle > 0$

Then, the average energy -second order moment of *f* - becomes,

$$E_\alpha(\boldsymbol{r},t) = \int_{-\infty}^{+\infty} \frac{m_\alpha v^2}{2} f_\alpha(\boldsymbol{v},\ \boldsymbol{r},\ t) d\boldsymbol{v} = n_\alpha \left\langle \frac{m_\alpha}{2} v^2 \right\rangle = n_\alpha \frac{m_\alpha}{2}(u_\alpha^2 + \langle w^2 \rangle) = \rho_\alpha \frac{u_\alpha^2}{2} + E_{i\alpha}$$

$$\rho_\alpha(\boldsymbol{r},t) = m_\alpha n_\alpha \qquad \boxed{E_\alpha(\boldsymbol{r},t) = \rho_\alpha \frac{u_\alpha^2}{2} + E_{i\alpha}} \qquad E_{i\alpha} = \int_{-\infty}^{+\infty} \frac{m_\alpha w^2}{2} f_\alpha(\boldsymbol{v},\ \boldsymbol{r},\ t) d\boldsymbol{w}$$

$$\boxed{\frac{3}{2} n_\alpha(\boldsymbol{r},t) k_B T_\alpha(\boldsymbol{r},t) = \int_{-\infty}^{+\infty} \left(\frac{m_\alpha w^2}{2}\right) f_\alpha(\boldsymbol{v},\ \boldsymbol{r},\ t) d\boldsymbol{w}}$$

This expressions imply the choice of A MODEL for the distribution function:
To solve , even in simple case, the kinetic equation is , a priori, imperative !

Since the zero, one and two order moments of *f* (density, momentum and energy) are the most relevant quantities to describe a plasma, a question arises:

Can one derive a set of evolution equations for these distribution moments (macroscopic fluid properties) to obtain them without solving the Kinetic equation?

Yes, It is possible, in fact, to derive such a set of fluid equations for the evolution of each plasma species macroscopic property.

Moreover: plasma fluid description is possible by fluid equations (each species is governed by its own fluid equation) . The equations are coupled to the Maxwell laws.

Some approximations are also possible, the most important approach gives the simple SINGLE FLUID plasma description or Magneto-hydrodynamics (MHD) .

Important remark: Each fluid approximation is *only valid* for a given specific plasma scenario characterized by typical lengths and time scales.

Basic idea: to directly extract from the Boltzmann equation a relation for the time rate of change of a given f-moment.

Is it simple? A problem arises due to the transport term $\frac{\partial}{\partial \mathbf{r}} \bullet \mathbf{v} f_{\alpha}$

The macroscopic fluid equations are deduced by taking moments of f and with Boltzmann Eq. The main function to be computed is the general k-order moment $M^k(\boldsymbol{r},t)$ expressed as the k-range tensor:

$$\boldsymbol{M}^{(k)}(\boldsymbol{r},t)=\int_{-\infty}^{+\infty}\underbrace{(\boldsymbol{v}\otimes\boldsymbol{v}\otimes\ldots\otimes\boldsymbol{v})}_{\text{k times}}g(\boldsymbol{v},\boldsymbol{r},t)d\boldsymbol{v}$$ where $g(\boldsymbol{v},\boldsymbol{r},t)$

is a distribution function. These moments are related to the averages, over the velocity space, of physical quantities; for a function $H_\alpha(v, r, t)$, the corresponding mean is $\langle H_\alpha \rangle$, an *explicit function of* r coordinates and t :

$$<H_\alpha(\boldsymbol{v},\boldsymbol{r},t)>=\int_{-\infty}^{+\infty}H_\alpha(\boldsymbol{v},\boldsymbol{r},t)\,\hat{f}_\alpha(\boldsymbol{v},\boldsymbol{r},t)d\boldsymbol{v}$$

$$<H_\alpha(\boldsymbol{v},\boldsymbol{r},t)>=\frac{1}{n_\alpha}\int_{-\infty}^{+\infty}H_\alpha(\boldsymbol{v},\boldsymbol{r},t)\,f_\alpha(\boldsymbol{v},\boldsymbol{r},t)d\boldsymbol{v}$$

Two equivalent definitions of averages.

For example, setting $H_\alpha(v, r, t) = 1$

$$n_\alpha(\boldsymbol{r},t)=\int_{-\infty}^{+\infty}f_\alpha(\boldsymbol{v},\boldsymbol{r},t)d\boldsymbol{v}$$

and with $H_\alpha(v, r, t) = v$

$$d\Gamma_\alpha(\boldsymbol{r},t)=n_\alpha(\boldsymbol{r},t)\,\boldsymbol{u}_\alpha(\boldsymbol{r},t)=\int_{-\infty}^{+\infty}\boldsymbol{v}\,f_\alpha(\boldsymbol{v},\boldsymbol{r},t)d\boldsymbol{v}$$

A general moment evolution equation (transport equation) can be easily obtained. For any scalar function H(v)

$$\langle H \rangle(r,t) = \frac{1}{n_\alpha(\mathbf{r},t)} \int H(\mathbf{v}) f_\alpha(\mathbf{r},\mathbf{v},t)\ d\mathbf{v} < \infty$$

Multiplying by H both sides of the kinetic equation and integrating over velocity space :

$$\int d\mathbf{v}\left[\frac{\partial f_\alpha}{\partial t} H + \frac{\partial}{\partial \mathbf{r}}\cdot\left(\mathbf{v}\ f_\alpha\ H\right) + \frac{q_\alpha}{m_\beta} H \frac{\partial}{\partial \mathbf{v}}\cdot\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right) f_\alpha = \left(\frac{\partial f_\alpha}{\partial t}\right)_c\right]$$

After integration, with Gauss Th. for the velocity divergence term

$$\int d\mathbf{v}\left[H \frac{\partial}{\partial \mathbf{v}}\cdot\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right) f_\alpha\right] =$$

$$\int d\mathbf{v} \frac{\partial}{\partial \mathbf{v}}\cdot\left[\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right) H\ f_\alpha\right] - \int d\mathbf{v}\ f_\alpha \left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\cdot \frac{\partial H}{\partial \mathbf{v}}$$

$$\text{with } \int d\mathbf{v} \frac{\partial}{\partial \mathbf{v}}\cdot\left[\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right) H\ f_\alpha\right] = \oint d\mathbf{S}_v \cdot\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right) H\ f_\alpha = 0$$

assuming that |v H f| decays to zero faster than $1/v^3$

So, the final time evolution of the H average is finally: (giving an effective and very general fluid equation !!!!)

$$\frac{\partial}{\partial t} n_\alpha \langle H \rangle + \frac{\partial}{\partial \mathbf{r}} \bullet \left[n_\alpha \langle \mathbf{v}\ H \rangle \right] - \frac{q_\alpha}{m_\beta} n_\alpha \left\langle (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \bullet \frac{\partial H}{\partial \mathbf{v}} \right\rangle =$$

$$\int d\mathbf{v} \left[H \left(\frac{\partial f_\alpha}{\partial t} \right)_c \right] \quad ; \text{ for } \quad H = 1,\ v_i,\quad v_i v_j,\quad v_i v_j v_k,\ \ldots\ (i, j, k = x, y, z)$$

A set of scalar, vector or tensor equations can be constructed from this. For example , for the second order moment:

$$\begin{aligned} \langle v_i v_j \rangle &= \langle (c_i^\alpha + u_i^\alpha)(c_j^\alpha + u_j^\alpha) \rangle \\ &= \langle c_i^\alpha c_j^\alpha \rangle + u_i^\alpha u_j^\alpha \\ &= \frac{T_\alpha}{m_\alpha} \delta_{ij} + \frac{\Pi_{ij}^\alpha}{m_\alpha n_\alpha} + u_i^\alpha u_j^\alpha \end{aligned}$$

(here, $\mathbf{c}_\alpha = \mathbf{v} - \mathbf{u}_\alpha = \mathbf{c}^\alpha = \mathbf{v} - \mathbf{u}^\alpha$)

Remarks (drawbacks of the fluid description):

1.- Any <H> evolution depends on spatial variations of the higher order moment < v H > (this is, the **transport of H** property itself !!): **an unclosed hierarchy of equations appears.**

2.- A **collisional model is still needed** to evaluate the second hand terms for every moment , in particular in must satisfy the properies:

zero net balance of number density, momentum and energy for each species, so it has to verify:

$$\int d\mathbf{v} \left(\frac{\partial f_\alpha}{\partial t}\right)_c = 0 \qquad (\alpha = \mathrm{i}, \mathrm{e})$$

$$\sum_\alpha m_\alpha \int d\mathbf{v}\, \mathbf{v} \left(\frac{\partial f_\alpha}{\partial t}\right)_c = 0$$

$$\sum_\alpha \frac{m_\alpha}{2} \int d\mathbf{v}\, v^2 \left(\frac{\partial f_\alpha}{\partial t}\right)_c = 0$$

In particular, in absence of source-sink "collisonal terms" continuity, and momentum transfer equations for each plasma species read:

$$\frac{\partial n_\alpha}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot (n_\alpha \mathbf{u}^\alpha) = 0$$

$$\frac{\partial}{\partial t}(m_\alpha n_\alpha u_i^\alpha) + \frac{\partial}{\partial r_i}(m_\alpha n_\alpha u_i^\alpha u_j^\alpha + n_\alpha T_\alpha \delta_{ij} + \Pi_{ij}^\alpha) - e_\alpha n_\alpha (E_i + (\mathbf{u}^\alpha \times \mathbf{B})_i) = R_i^\alpha$$

With the rate of momentum transferred to species α form the others as:

$$\mathbf{R}^\alpha = m_\alpha \int d\mathbf{v}\, \mathbf{v} \left(\frac{\partial f_\alpha}{\partial t}\right)_c$$

and the momentum equation involves unknown **second order** moment (kinetic energy and stress tensor).

General formulation :

The **second order** moments are related to the flux of momentum (here we use the local stress tensor, for general stress tensor v changes into v-w)

$$\tilde{\boldsymbol{H}}_\alpha(\boldsymbol{v},\boldsymbol{r},t)=m_\alpha\,(\boldsymbol{v}\otimes\boldsymbol{v}) \qquad \tilde{\boldsymbol{S}}_\alpha(\boldsymbol{r},t)=\int_{-\infty}^{+\infty} m_\alpha(\boldsymbol{v}\otimes\boldsymbol{v})f_\alpha(\boldsymbol{v},\boldsymbol{r},t)d\boldsymbol{v}$$

The **kinetic energy** (internal energy), with $(\mathbf{v}\otimes\mathbf{v})_{ij}=v_i\,v_j$

$$H_\alpha(\boldsymbol{v},\boldsymbol{r},t)=\frac{m_\alpha}{2}\,(\boldsymbol{v}\bullet\boldsymbol{v})=\frac{m_\alpha v^2}{2} \qquad E_\alpha(\boldsymbol{r},t)=\int_{-\infty}^{+\infty}\frac{m_\alpha v^2}{2}f_\alpha(\boldsymbol{v},\boldsymbol{r},t)d\boldsymbol{v}$$

The flux of kinetic energy **(heat flux, in fact,with three 3rd order moments responsible of the transport of kinetic energy),**

$$H_\alpha(\boldsymbol{v},\boldsymbol{r},t)=\frac{m_\alpha}{2}\,(\boldsymbol{v}\bullet\boldsymbol{v})\,\boldsymbol{v} \qquad \boldsymbol{K}_\alpha(\boldsymbol{r},t)=\int_{-\infty}^{+\infty}(\frac{m_\alpha v^2}{2})\,\boldsymbol{v}\,f_\alpha(\boldsymbol{v},\boldsymbol{r},t)d\boldsymbol{v}$$

The velocity distribution function is not determined but a justified model for it can be used. At this point we require a **bounded well-behaved distribution function model** to compute the k-order moment such that:

$$v^{k+3}f_\alpha(\mathbf{v},\mathbf{r},t)<\frac{1}{v} \qquad \text{for} \qquad v\rightarrow\infty$$

Only the first 3 moments of *f* are more physically relevant *but* they are "transported" by higher order moments evolution !!!!

The local stress symmetric tensor:

$$\tilde{\boldsymbol{S}}_\alpha(\boldsymbol{r},t)=\int_{-\infty}^{+\infty} m_\alpha(\boldsymbol{v}\otimes\boldsymbol{v}) f_\alpha(\boldsymbol{v},\,\boldsymbol{r},t)\,d\boldsymbol{v}=m_\alpha n_\alpha(\boldsymbol{r},t)\langle\boldsymbol{v}\otimes\boldsymbol{v}\rangle$$

the components are:

That can be measured in terms of **the random - thermal- velocity**

$$\begin{pmatrix} v_x^2 & v_x v_y & v_x v_z \\ v_y v_x & v_y^2 & v_y v_z \\ v_z v_x & v_z v_y & v_z^2 \end{pmatrix}$$

$$\left\langle (u_{\alpha i}+w_i)(u_{\alpha j}+w_j)\right\rangle$$

$$\boldsymbol{v}=\boldsymbol{u}_\alpha+\boldsymbol{w}$$

$$S_{ij}=m_\alpha n_\alpha(\langle u_{\alpha i}u_{\alpha j}\rangle+\langle w_i w_j\rangle+u_{\alpha i}\langle w_j\rangle+u_{\alpha j}\langle w_i\rangle)$$

$$\tilde{\boldsymbol{S}}_\alpha=m_\alpha n_\alpha(\boldsymbol{u}_\alpha\otimes\boldsymbol{u}_\alpha)+m_\alpha n_\alpha\langle\boldsymbol{w}\otimes\boldsymbol{w}\rangle$$

$$\boxed{\tilde{\boldsymbol{G}}_\alpha=m_\alpha n_\alpha\langle\boldsymbol{w}\otimes\boldsymbol{w}\rangle=\boldsymbol{P}_\alpha+\boldsymbol{\Pi}_\alpha}$$

$$\tilde{\boldsymbol{P}}_\alpha=m_\alpha n_\alpha\begin{pmatrix} < w_x^2 > & 0 & 0 \\ 0 & < w_y^2 > & 0 \\ 0 & 0 & < w_y^2 > \end{pmatrix}$$

$$\tilde{\Pi}_\alpha=m_\alpha n_\alpha\begin{pmatrix} 0 & < w_x w_y > & < w_x w_z > \\ < w_y w_x > & 0 & < w_y w_z > \\ < w_z w_x > & < w_z w_y > & 0 \end{pmatrix}$$

diagonal tensor
(anisotropic scalar pressure)

Off-diagonal Symmetric tensor
(stresses)

This defines the scalar pressure -*isotropic case* if no privileged direction is held inside the plasma-

$$\langle w_x^2 \rangle = \langle w_x^2 \rangle = \langle w_x^2 \rangle = \frac{\langle w^2 \rangle}{3} \qquad p_\alpha = \frac{1}{3} Tr(\tilde{P}_\alpha)$$

$$p_\alpha(\boldsymbol{r},t) = \frac{m_\alpha}{3} \int_{-\infty}^{+\infty} w^2 f_\alpha(\boldsymbol{v},\boldsymbol{r},t) d\boldsymbol{v} = \frac{m_\alpha n_\alpha}{3} < w^2 >$$

and the symmetric stress tensor becomes

$$\begin{cases} \tilde{\boldsymbol{P}}_\alpha = p_\alpha(\boldsymbol{r},t)\, \tilde{\boldsymbol{I}} \\ p_\alpha(\boldsymbol{r},t) = n_\alpha(\boldsymbol{r},t)\; k_B T_\alpha(\boldsymbol{r},t) \end{cases}$$

NOTE again that this choice implies A MODEL (or the knowledge) for the distribution function !

The <u>energy transport</u> is related to

$$\boldsymbol{K}_\alpha(\boldsymbol{r},t) = \int_{-\infty}^{+\infty} (\frac{m_\alpha v^2}{2}) \boldsymbol{v} f_\alpha(\boldsymbol{v},\, \boldsymbol{r},\, t) d\boldsymbol{v} = \frac{m_\alpha n_\alpha}{2} < v^2 \boldsymbol{v} >$$

and finally we obtain the vector,

$$\boldsymbol{K}_\alpha(\boldsymbol{r},t) = E_\alpha \boldsymbol{u}_\alpha + \boldsymbol{q}_\alpha + \tilde{\mathbf{G}}_\alpha : \boldsymbol{u}_\alpha$$

$$\boldsymbol{q}_\alpha(\boldsymbol{r},\, t) = \int_{-\infty}^{+\infty} (\frac{m_\alpha w^2}{2}) \boldsymbol{w} f_\alpha(\boldsymbol{v},\, \boldsymbol{r},\, t) d\boldsymbol{v} = \frac{m_\alpha n_\alpha}{2} < w^2 \boldsymbol{w} >$$

Where **q** is the heat flux density,

Note:

Plasma spatial anisotropy emerges naturally when fields E and B appears (externally imposed fields and/or coming from collective effects, also density n and temperature T **gradients** lead to anisotropies and huge transport processes)

E.g. In a strong magnetic field, plasma species can **be magnetized**, thus the **pressure tensor becomes anisotropic, plasma is** far from equilibrium, it is convenient to distinguish two dynamics, **perpendicular and parallel** to ***B***, we rewrite (very usual in space plasmas):

$$\mathbf{w} = \mathrm{w}_{\parallel} \frac{\mathbf{B}}{B} + \mathbf{w}_{\perp} = \mathrm{w}_{\parallel} \mathbf{b} + \mathbf{w}_{\perp}$$

$$\langle \mathbf{w} \otimes \mathbf{w} \rangle = \frac{T_{\perp}}{m} \mathbf{1} + \frac{1}{m} \left(T_{\parallel} - T_{\perp} \right) \mathbf{b} \otimes \mathbf{b}$$

For each magnetized plasma species: collision frequency much lower than gyro-frequency in magnetic field.
In the opposite case,collisions isopropizes the system, so
Perpendicular and parallel temperatures become equal at equilibrium

The *same averaging* procedure is exended if source-sink terms exists ... E.g.

$$\int_{-\infty}^{+\infty} \frac{\partial f_\alpha}{\partial t} d\boldsymbol{v} + \int_{-\infty}^{+\infty} \nabla_r \cdot (\boldsymbol{v} f_\alpha) d\boldsymbol{v} + \int_{-\infty}^{+\infty} \nabla_v \cdot (\frac{\boldsymbol{F}}{m_\alpha} f_\alpha) d\boldsymbol{v} = \int_{-\infty}^{+\infty} C_s(f_\alpha) d\boldsymbol{v}$$

$$\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} f_\alpha d\boldsymbol{v} + \nabla_r \cdot \int_{-\infty}^{+\infty} (\boldsymbol{v} f_\alpha) d\boldsymbol{v} + \underbrace{\int_{s(v)} (\frac{\boldsymbol{F}}{m_\alpha} f_\alpha) \cdot d\boldsymbol{S}}_{f_\alpha \text{ is bounded} \rightarrow 0} = \int_{-\infty}^{+\infty} C_s(f_\alpha) d\boldsymbol{v}$$

$$\boxed{\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \boldsymbol{u}_\alpha) = \int_{-\infty}^{+\infty} C_s(f_\alpha) d\boldsymbol{v} = F_\alpha - S_\alpha}$$

First moment of the Boltzmann collision operator.

Sources of a particles

Sinks of a particles

The averages consider an unspecified, bounded distribution function.

The fluid transport equations for a plasma, ...

From the three moments of the Botzmann equation, are deduced the transport of particles momentum and energy

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \boldsymbol{u}_\alpha) = F_\alpha - S_\alpha$$

$$\rho_\alpha \frac{D\,\boldsymbol{u}_\alpha}{Dt} = -\nabla_{\boldsymbol{r}}\, p_\alpha - \nabla_{\boldsymbol{r}} \cdot \tilde{\Pi}_\alpha - m_\alpha \boldsymbol{u}_\alpha (F_\alpha - G_\alpha) + \boldsymbol{F}_\alpha^e + \boldsymbol{F}_\alpha^g + \boldsymbol{R}_\alpha$$

$$E_\alpha = \rho_\alpha (u_\alpha^2 / 2 + e_{i\alpha})$$

$$\frac{DE_\alpha}{Dt} = -\nabla_{\boldsymbol{r}} \cdot (p_\alpha\ \boldsymbol{u}_\alpha) + \boldsymbol{J}_\alpha \cdot \boldsymbol{E} - \nabla_{\boldsymbol{r}} \cdot \boldsymbol{q}_\alpha - \nabla_{\boldsymbol{r}} \cdot (\tilde{\Pi}_\alpha : \boldsymbol{u}_\alpha) + Q_\alpha$$

$$F_\alpha - S_\alpha = \int_{-\infty}^{+\infty} C_s(f_\alpha)\, d\boldsymbol{v}$$

$$\boldsymbol{R}_\alpha = \int_{-\infty}^{+\infty} m_\alpha \boldsymbol{v}\, C_s(f_\alpha)\ d\boldsymbol{v} \qquad Q_\alpha = \int_{-\infty}^{+\infty} \frac{m_\alpha v^2}{2} C_s(f_\alpha)\, d\boldsymbol{v}$$

For an exhaustive and compactly given system of fluid equations, useful in space plasmas, see the book :

Ionospheres: Physics, Plasma Physics, and Chemistry, RW. Schunk, A F Nagy. Cambridge 2000 .

Where different approximations of distribution functions can be found, from the one-order moment isotropic Maxwellian to the 13-order moments approximation distribution, in appendix H

For a simple approach to the single fluid plasmadynamic equations, named MHD equations, coupled to reduced Maxwell equations see, for instance:

Principles of Plasma Physics for Engineers and Scientists, U S Inan and M Gołkowski, Cambridge University Press, 2011

Example:

fluid equations for space plasmas accounting for source-sink terms and ionization processes. All terms cam be modeled by appropriate (experimental) frequencies υ

$$\begin{cases} \dfrac{\partial n_\alpha}{\partial t} + \nabla\cdot(n_\alpha \boldsymbol{u}_\alpha) = F_\alpha - S_\alpha \\ \rho_\alpha \dfrac{D\,\boldsymbol{u}_\alpha}{Dt} = -\nabla_{\boldsymbol{r}}\, p_\alpha - m_\alpha \boldsymbol{u}_\alpha (F_\alpha - G_\alpha) + \boldsymbol{F}^e_\alpha + \boldsymbol{F}^g_\alpha + \boldsymbol{R}_\alpha \\ \dfrac{DE_\alpha}{Dt} = -\nabla_{\boldsymbol{r}}\cdot(p_\alpha\,\boldsymbol{u}_\alpha) + \boldsymbol{J}_\alpha\cdot\boldsymbol{E} - \nabla_{\boldsymbol{r}}\cdot\boldsymbol{q}_\alpha + Q_\alpha \end{cases}$$

$$(F_e = \nu_I\, n_e)$$ ...etc.

$$E_\alpha = \rho_\alpha (u_\alpha^2/2 + e_{i\alpha})$$

For elastic collisions:

$$\boldsymbol{R}_\alpha = \sum_\beta \boldsymbol{R}_{\alpha\beta} \qquad \boldsymbol{R}_{\alpha\beta} = -A_{\alpha\beta}(\boldsymbol{u}_\alpha - \boldsymbol{u}_\beta)$$

$$\boldsymbol{R}_{\alpha\beta} = -\boldsymbol{R}_{\alpha\beta} \qquad \sum_\alpha \boldsymbol{R}_\alpha = \sum_\alpha\sum_\beta \boldsymbol{R}_{\alpha\beta} = 0$$

$$\begin{cases} A_{\alpha\beta} = n_\alpha n_\beta\, \mu_{\alpha\beta}\sigma_{\alpha\beta}\dfrac{4}{3}\left(\dfrac{8k_BT}{\pi\,\mu_{\alpha\beta}}\right)^{1/2} \\ A_{ea} \cong n_e n_a\, m_e\, \sigma_{ea}\dfrac{4\sqrt{2}}{3}\left(\dfrac{2k_BT_e}{\pi m_e}\right)^{1/2} \end{cases}$$

Energy:

$$Q_\alpha = \sum_\beta Q_{\alpha\beta}$$

$$Q_{tot} = Q_{\alpha\beta} + Q_{\beta\alpha} = -\boldsymbol{R}_{\alpha\beta}\cdot\boldsymbol{u}_\alpha - \boldsymbol{R}_{\beta\alpha}\cdot\boldsymbol{u}_\beta = -\boldsymbol{R}_{\alpha\beta}\cdot(\boldsymbol{u}_\alpha - \boldsymbol{u}_\beta) > 0$$

For ionizing collisions

$$\boldsymbol{R}_{Ii} = -m_i n_e \nu_I(\boldsymbol{u}_i - \boldsymbol{u}_e) \qquad \boldsymbol{R}_{Ie} = -m_i n_e \nu_I(\boldsymbol{u}_e - \boldsymbol{u}_i) \qquad Q_I = E_i\,\nu_I$$

bulk drift velocity	$\boldsymbol{u}_\mathrm{i} = \langle \boldsymbol{v}_\mathrm{i} \rangle$
pressure tensor	$\boldsymbol{p}_\mathrm{i} = n_\mathrm{i} m_\mathrm{i} \langle \boldsymbol{c}_\mathrm{i} \boldsymbol{c}_\mathrm{i} \rangle$
pressure	$p_\mathrm{i} = 1/3 n_\mathrm{i} m_\mathrm{i} \langle c_\mathrm{i}^2 \rangle$
stress tensor	$\boldsymbol{\tau}_\mathrm{i} = \boldsymbol{P}_\mathrm{i} - p_\mathrm{i} \boldsymbol{I}$
heat flow tensor	$\boldsymbol{Q}_\mathrm{i} = n_\mathrm{i} m_\mathrm{i} \langle \boldsymbol{c}_\mathrm{i} \boldsymbol{c}_\mathrm{i} \boldsymbol{c}_\mathrm{i} \rangle$
and heat flow vector	$\boldsymbol{q}_\mathrm{i} = 1/2 n_\mathrm{i} m_\mathrm{i} \langle c_\mathrm{i}^2 \boldsymbol{c}_\mathrm{i} \rangle$.

$$\boldsymbol{c}_\mathrm{i} = \boldsymbol{v}_\mathrm{i} - \boldsymbol{u}_\mathrm{i}$$

$$f_\mathrm{i}^{(0)} = n_\mathrm{i} \left(\frac{m_\mathrm{i}}{2\pi k_\mathrm{b} T_\mathrm{i}} \right)^{\frac{3}{2}} \exp\left(-\frac{m_\mathrm{i} c_\mathrm{i}^2}{2 k_\mathrm{b} T_i} \right)$$

13-moment approximation takes the form

$$f_\mathrm{i} = f_\mathrm{i}^{(0)} \left[1 + \frac{m_\mathrm{i}}{2 k_\mathrm{b} T_\mathrm{i} p_\mathrm{i}} \boldsymbol{\tau}_\mathrm{i} : \boldsymbol{c}_\mathrm{i} \boldsymbol{c}_\mathrm{i} - \left(1 - \frac{m_\mathrm{i} c_\mathrm{i}^2}{5 k_\mathrm{b} T_\mathrm{i}} \right) \frac{m_\mathrm{i}}{k_\mathrm{b} T_\mathrm{i} p_\mathrm{i}} \boldsymbol{q}_\mathrm{i} \boldsymbol{c}_\mathrm{i} \right]$$

20-moment approximation

$$f_\mathrm{i} = f_\mathrm{i}^{(0)} \left[1 + \frac{m_\mathrm{i}}{2 k_\mathrm{b} T_\mathrm{i} p_\mathrm{i}} \boldsymbol{\tau}_\mathrm{i} : \boldsymbol{c}_\mathrm{i} \boldsymbol{c}_\mathrm{i} + \frac{m_\mathrm{i}}{2 k_\mathrm{b}^2 T_\mathrm{i}^2 p_\mathrm{i}} \boldsymbol{Q}_\mathrm{i} \vdots \boldsymbol{c}_\mathrm{i} \boldsymbol{c}_\mathrm{i} \boldsymbol{c}_\mathrm{i} - \frac{m_\mathrm{i}}{k_\mathrm{b} T_\mathrm{i} p_\mathrm{i}} \boldsymbol{q}_\mathrm{i} \boldsymbol{c}_\mathrm{i} \right]$$