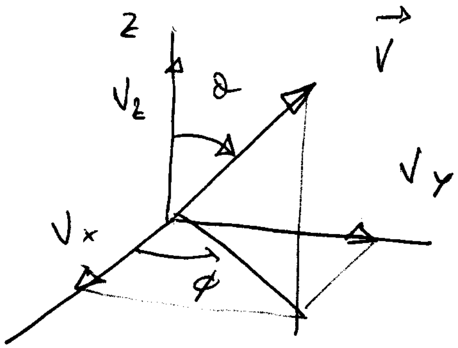
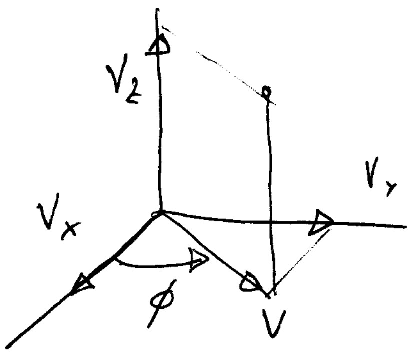
Calculo de las velocidades





coord. Esf.	coord. Cil.
$V_x = V \,\mathrm{sen}\,\theta \cos\phi$	$V_x = V \cos\phi$
$V_y = V \,\mathrm{sen}\,\theta \,\mathrm{sen}\,\phi$	$V_y = V \,\mathrm{sen}\,\phi$
$V_z = V \cos\theta$	$V_z = V_z$
$d^3v = (v^2 \,\mathrm{sen}\,\theta)\, dv\, d\theta\, d\phi$	$d^3v = v\, dv\, d\theta\, dv_z$

$$\langle V_x \rangle = \int_0^{\infty} V_x f(v)\, d^3v = \int_{-\infty}^{\infty} V_x f(v)\, dv_x\, dv_y\, dv_z = \left(\frac{m}{2\pi KT}\right)^{3/2} \int_{-\infty}^{+\infty} V_x\, e^{-mv^2/2KT}\, dv_x\, dv_y\, dv_z$$

$$\langle V_x \rangle = \left(\frac{m}{2\pi KT}\right)^{3/2} \Big[\underbrace{\int_{-\infty}^{+\infty} V_x\, e^{-mV_x^2/2KT}\, dv_x}_{=\,0} \Big] \Big[\underbrace{\int_{-\infty}^{+\infty} e^{-mV_y^2/2KT}\, dv_y}_{=\,\left(\frac{2KT\pi}{m}\right)^{1/2}} \Big] \Big[\underbrace{\int_{-\infty}^{+\infty} e^{-mV_z^2/2KT}\, dv_z}_{=\,\left(\frac{2\pi KT}{m}\right)^{1/2}} \Big]$$

$$\langle V_x \rangle = 0$$

otro modo:

$$\langle V_x \rangle = \int_{-\infty}^{+\infty} f(v)\, v_x\, d^3v = \int_{-\infty}^{+\infty} v_x f(v)\, [v^2 \,\mathrm{sen}\,\theta]\, d\theta\, d\phi\, dv$$

$$\langle V_x \rangle = \int_{-\infty}^{+\infty} [\underbrace{V \,\mathrm{sen}\,\theta \cos\phi}_{=\,V_x}]\, [v^2 \,\mathrm{sen}\,\theta]\, d\theta\, d\phi\, dv$$

$0 \leq v < \infty$
$0 \leq \phi \leq 2\pi$
$0 \leq \theta \leq \pi$
Coordenadas esféricas.

$$\langle V_x \rangle = \left[\int_0^{\infty} v^3 f(v)\, dv \right] \times \left[\underbrace{\int_0^{2\pi} \cos\phi \, d\phi}_{=0} \right] \times \left[\underbrace{\int_0^{\pi} \operatorname{sen}^2\theta \, d\theta}_{=\pi/2} \right]$$

y como $\int_0^{2\pi} \cos\phi \, d\phi = \int_0^{2\pi} \operatorname{sen}\phi \, d\phi = 0$ y $\int_0^{\pi} \cos\theta \, d\theta = 0$ se tiene lo mismo para las otras componentes, luego $\langle V_x \rangle = \langle V_y \rangle = \langle V_z \rangle = 0$ y se obtiene el mismo resultado empleando las exponenciales.

Otra velocidad importante es la "velocidad promedio" $\langle v \rangle$

$$\langle v \rangle = \int_{-\infty}^{+\infty} v f(v)\, d^3v = \int_{-\infty}^{+\infty} v f(v) \left[v^2 \operatorname{sen}\theta \right] dv\, d\theta\, d\phi$$

$$\langle v \rangle = \left[\int_0^{\infty} v^3 f(v)\, dv \right] \times \left[\underbrace{\int_0^{2\pi} d\phi}_{=2\pi} \right] \times \left[\underbrace{\int_0^{\pi} \operatorname{sen}\theta \, d\theta}_{=2} \right] = 4\pi \int_0^{\infty} v^3 f(v)\, dv$$

Para simplificar: $V_{th} = \left(\frac{2KT}{m} \right)^{1/2}$ $\quad f(v) = \frac{1}{\pi^{3/2} V_{th}^3} e^{-v^2/V_{th}^2}$ otra forma de la distribución de MB

Con esto resulta:

$$\langle v\rangle = 4\pi \int_0^\infty \frac{1}{\pi^{3/2} v_{th}^3} v^3 e^{-v^2/v_{th}^2} dv = 4\pi \frac{v_{th}^4}{\pi^{3/2}} \int_0^\infty s^3 e^{-s^2} ds$$

$$\langle v\rangle = \frac{4 v_{th}^4}{\sqrt{\pi}} [\int_0^\infty \underbrace{s^3 e^{-s^2} ds}_{= 1/2}] = \frac{4 v_{th}^4}{\sqrt{\pi}} \times \frac{1}{2} = \frac{2}{\sqrt{\pi}} \left(\frac{2kT}{m}\right)^{1/2} = \left(\frac{8kT}{\pi m}\right)^{1/2}$$

En cambio $\langle |v_x| \rangle \neq 0$.

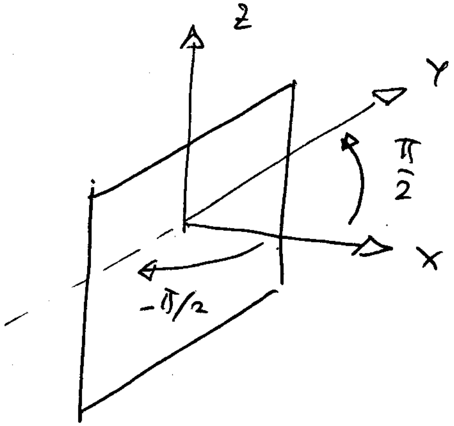
Si volvemos a la ecuación anterior:

$$\langle |v_x| \rangle = \int_{-\infty}^{\infty} |v_x| f(v) d^3v = \left(\frac{m}{2\pi kT}\right)^{3/2} [\int_{-\infty}^{\infty} |v_x| e^{-mv_x^2/2kT} dv_x] \times \underbrace{[\int_{-\infty}^{+\infty} e^{-\frac{mv_y^2}{2kT}} dv_y]}_{= \left(\frac{2kT\pi}{m}\right)} \times [\int_{-\infty}^{+\infty} e^{-\frac{mv_z^2}{2kT}} v_z]$$

$$\langle |v_x| \rangle = \left(\frac{m}{2\pi kT}\right)^{1/2} [\int_{-\infty}^{+\infty} |v_x| e^{-mv_x^2/2kT} dv_x]$$

$$\langle |v_x| \rangle = \frac{1}{\sqrt{\pi}\, v_{th}} v_{th}^2 [\int_{-\infty}^{+\infty} |s| e^{-s^2} ds] = \frac{v_{th}}{\sqrt{\pi}} = \left(\frac{2kT}{\pi m}\right)^{1/2}$$

$$\left(2\int_0^\infty s e^{-s^2} ds = 1 \right)$$



Para calcularlo de otro modo hay que interpretar los ángulos $|V_x|>0$ es contar 2 veces el ángulo ϕ entre $\pi/2$ y $-\pi/2$. Empleando la ec. anterior

$$\langle |V_x| \rangle = \left[\int_0^\infty v^3 f(v)\, dv\right] \times \underbrace{\left[2\int_{-\pi/2}^{+\pi/2} \cos\phi\, d\phi\right]}_{=2} \times \underbrace{\left[\int_0^\pi \operatorname{sen}^2\theta\, d\theta\right]}_{=\pi/2}$$

(?)

$$\langle |V_x| \rangle = 2\pi \int_0^\infty v^3 f(v)\, dv$$ (ojo con este 2)

$$\langle |V_x| \rangle = 2\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \int_0^\infty v^3 e^{-mv^2/2kT}\, dv$$ y con $s = v/v_{th}$

$$\langle |V_x| \rangle = 2\pi \frac{1}{\pi^{3/2} V_{th}^3} V_{th}^4 \underbrace{\int_0^\infty s^3 e^{-s^2}\, ds}_{=1/2} = 2\pi \frac{V_{th}^4}{\pi^{3/2} V_{th}^3} \times \frac{1}{2}$$

$$\langle |V_x| \rangle = \frac{V_{th}}{\sqrt{\pi}} = \left(\frac{2kT}{\pi m}\right)^{1/2}$$ Que es lo que obtuvimos antes.

Para algunos cálculos es mas cómodo emplear la energía

$$g(E) = \frac{2}{\sqrt{\pi}} \frac{\sqrt{E}}{(K_B T)^{3/2}} e^{-E/K_B T} \qquad \int_0^\infty g(E)\, dE = 1$$

Prop. de $\Gamma(n)$

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\, dt$$

$$\Gamma(n+1) = n\,\Gamma(n)$$

$$\Gamma(n+1) = n!$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\langle E \rangle = \int_0^\infty E\, g(E)\, dE = \frac{2}{\sqrt{\pi}} \frac{1}{(K_B T)^{3/2}} \int_0^\infty E^{3/2} e^{-E/K T}\, dE$$

$$\langle E \rangle = \frac{2}{\sqrt{\pi}} \frac{1}{(K_B T)^{3/2}} (K_B T)^{\frac{3}{2}+1} \underbrace{\int_0^\infty s^{3/2} e^{-s}\, ds}_{} = \quad \Gamma\left(\tfrac{5}{2}\right) = \frac{3}{2}\Gamma\left(\tfrac{3}{2}\right) = \frac{3}{2} \times \frac{1}{2} \Gamma\left(\tfrac{1}{2}\right)$$

$$\langle E \rangle = \frac{2}{\sqrt{\pi}} \frac{(K_B T)^{5/2}}{(K_B T)^{3/2}} \times \frac{3}{4}\sqrt{\pi} = \frac{3}{2} K_B T$$

Que es el cálculo anterior.

como $v = \left(\frac{2E}{m}\right)^{1/2}$ podemos repetir de otro modo el cálculo anterior.

$$\left\langle \left(\frac{2E}{m}\right)^{1/2} \right\rangle = \langle v \rangle = \int_0^\infty \left(\frac{2E}{m}\right)^{1/2} g(E)\, dE = \sqrt{\frac{2}{m}} \int_0^\infty \sqrt{E}\, g(E)\, dE$$

6

$$\langle v \rangle = \left\langle \sqrt{\frac{2E}{m}} \right\rangle = \int_0^\infty \left(\frac{2E}{m}\right)^{1/2} g(E)\,dE = \sqrt{\frac{2}{m}} \int_0^\infty \sqrt{E}\, g(E)\,dE$$

$$\langle v \rangle = \sqrt{\frac{2}{m}}\,\frac{2}{\sqrt{\pi}} \int_0^\infty E\,\frac{e^{-E/kT}}{(k_BT)^{3/2}}\,dE = \sqrt{\frac{8}{\pi m}}\,\frac{1}{(k_BT)^{3/2}} \int_0^\infty E\,e^{-E/kT}\,dE$$

$$\langle v \rangle = \sqrt{\frac{8}{\pi m}}\,\frac{1}{(k_BT)^{3/2}}\,(k_BT)^2 \int_0^\infty \underbrace{s\,e^{-s}\,ds}_{\Gamma(1) = 1!} = \left(\frac{8}{\pi m}\right)^{1/2} (k_BT)^{2-\frac{3}{2}}$$

$$\langle v \rangle = \left(\frac{8kT}{\pi m}\right)^{1/2}$$ Que es el resultado anterior.