Debye length

(1)

$$\delta\rho = \delta\rho_{pl} + \delta\rho_L \qquad \begin{cases} \delta\rho_L = q\,\delta(\bar r) \\ \delta\rho_{pl} = e\,[n_i(\bar r) - n_e(\bar r)] \end{cases}$$

q small $\quad |\phi(\bar r)| \ll 1 \quad \bar r = 0$

$\phi = \varphi - \varphi_0 \quad \varphi_0$ = uniform

$n_{e0} \simeq n_{i0} \simeq 0 \quad$ equilibrium

quasineutrality plasma potential

$$\left.\begin{aligned} -\nabla^2\varphi &= \rho/\epsilon_0 \\ \bar E &= -\nabla\varphi \end{aligned}\right\} \quad -\nabla^2\varphi = \frac{\rho}{\epsilon_0} \qquad -\nabla^2\varphi = \frac{\delta\rho_{pl} + \delta\rho_L}{\epsilon_0}$$

$$-\nabla^2\varphi = \frac{1}{\epsilon_0}\{q\,\delta(\bar r)\} + \frac{e}{\epsilon_0}[n_i(\bar r) - n_e(\bar r)]$$

$$\begin{cases} n_i(\bar r) \simeq n_{i0}\left(1 - \dfrac{e\varphi}{K_B T_i}\right) \\ n_e(x) \simeq n_{e0}\left(1 + \dfrac{e\varphi}{K_B T_e}\right) \end{cases}$$

$$-\nabla^2\varphi = \frac{q}{\epsilon_0}\delta(\bar r) + \frac{e n_0}{\epsilon_0}\left[1 - \frac{e\varphi}{K_B T_i} - 1 - \frac{e\varphi}{K_B T_e}\right]$$

Approximation only valid for $\left|\frac{e\phi}{KT}\right| \ll 1$

$$-\nabla^2\varphi = \frac{q}{\epsilon_0}\delta(\bar r) + \varphi\left[\frac{e^2 n_0}{K_B T_i \epsilon_0} + \frac{e^2 n_0}{K_B T_e \epsilon_0}\right]$$

$$\begin{cases} \lambda_{De} = \left(\dfrac{K_B T_e \epsilon_0}{e^2 n_0}\right)^{1/2} \\ \lambda_{Di} = \left(\dfrac{K_B T_i \epsilon_0}{e^2 n_0}\right)^{1/2} \end{cases}$$

$$\left[\nabla^2 - \frac{1}{\Lambda_D^2}\right]\varphi = -\frac{q}{\epsilon_0}\delta(\bar r) \rightarrow \frac{1}{\Lambda_D^2} = \frac{1}{\lambda_{De}^2} + \frac{1}{\lambda_{Di}^2}$$

Spherical symmetry $\left.\right\}$ $\dfrac{\partial^2\varphi}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial\varphi}{\partial r} - \dfrac{\varphi}{\Lambda_D^2} = -\dfrac{q}{\epsilon_0}\delta(r) \rightarrow$ solutions $\phi = \varphi - \varphi_0 \quad \phi = \dfrac{A}{r} f(r)$

$A > 0$

$$\frac{\partial \phi}{\partial r} = A\left[\frac{1}{r}\frac{\partial f}{dr} - \frac{1}{r^2} f(r)\right]$$

$$\frac{\partial^2 \phi}{\partial r^2} = A\left[-\frac{1}{r^2}\frac{df}{dr} + \frac{1}{r}\frac{d^2 f}{dr^2} + \frac{2}{r^3} f(r) - \frac{1}{r^2}\frac{df}{dr}\right]$$

$$\frac{\partial^2 \phi}{\partial r^2} = A\left[\frac{1}{r}\frac{d^2 f}{dr^2} - \frac{2}{r^2}\frac{df}{dr} + \frac{2}{r^3} f(r)\right]$$

$$\overbrace{A\left[\frac{1}{r}\frac{d^2 f}{dr^2} - \frac{2}{r^2}\frac{df}{dr} + \frac{2}{r^3} f(r)\right]}^{\partial^2\phi/\partial r^2} + \frac{2A}{r}\overbrace{\left[\frac{1}{r}\frac{df}{dr} - \frac{1}{r^2} f(r)\right]}^{\partial\phi/\partial r} - \frac{1}{\Lambda_D^2}\frac{A}{r} f(r) = -\frac{q\,\delta(r)}{\varepsilon_0}$$

$$A\left[\frac{1}{r}\frac{d^2 f}{dr^2} - \frac{2}{r^2}\frac{df}{dr} + \frac{2}{r^3} f(r) + \frac{2}{r^2}\frac{df}{dr} - \frac{2}{r^3} f(r)\right] - \frac{A}{\Lambda_D^2}\frac{f(r)}{r} = -\frac{q}{\varepsilon_0}\delta(\bar{r})$$

$$\frac{A}{r}\frac{d^2 f}{dr^2} - \frac{A}{r} f(r) = -\frac{q}{\varepsilon_0}\delta(\bar{r}) \Rightarrow \frac{A}{r}\left[\frac{d^2 f}{dr^2} - \frac{1}{\Lambda_D^2}\right] = -\frac{q}{\varepsilon_0}\delta(\bar{r})$$

For $r > 0$ $\quad \dfrac{d^2 f}{dr^2} - \dfrac{1}{\Lambda^2} = 0 \Rightarrow f(r) = e^{\pm r/\Lambda_D}$

$\oplus$ No physical meaning

$\ominus$ exponential decrease

Solutions are $\quad \phi(r) = \frac{A}{r} e^{-r/\Lambda_D}$ (3)

the physical meaning of A is obtained in the limit $|r/\Lambda_D| \ll 1$ where

$$\phi(r) \sim \frac{A}{r}\left(1 - \frac{r}{\Lambda_D}\right) \rightarrow A = \frac{q}{4\pi\epsilon_0}$$ recovers isolated electric charge

And hence

$$\phi(r) = \frac{q}{4\pi\epsilon_0} \frac{e^{-r/\Lambda}}{r}$$ $\rightarrow$ Physical meaning: shielding