Some Collision models

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The Boltzmann equation for distribution functions depends on the form of the collisional term C(f)

$$\frac{\partial f_\alpha}{\partial t} + \nabla_r \cdot (\boldsymbol{v} f_\alpha) + \nabla_v \cdot (\frac{\boldsymbol{F}}{m_\alpha} f_\alpha) = C(f_\alpha) \qquad f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t) \quad \alpha = e, i, a$$

C(f) be simplified by using somewhat simple models.

The Krook (Bathnagar-Gross-Krook) model is one of the simplest C(f).

(Bhatnagar, P.L., Gross, E.P. and Krook, M. Phys. Rev. ***94, 511****(1954))*

If a plasma is close to the isotropic thermal equilibrium, i.e. , close to the **local Maxwellian** *fo*, the effects of binary collisions can be modeled by means of a collisional frequency as : (species indexes are suppressed)

$$\left(\frac{\partial f}{\partial t}\right)_c = -\nu_c (f - f_0)$$

$$f_0(\mathbf{r}, \mathbf{v}) = n(\mathbf{r}) \left(\frac{m}{2\pi k_B T(\mathbf{r})}\right)^{3/2} \exp\left\{-m[\mathbf{v} - \mathbf{u}(\mathbf{r})]^2 / 2k_B T(\mathbf{r})\right\}$$

for a single species plasma (e.g. electrons in a neutralizing background of ions at rest). Extension to several species is straightforward.

The system characteristic **relaxation time** is given by the inverse of the frequency ν_c, this can be seen from the solution for constant frequency ν_0 :

$$f_s(\mathbf{v}, t) = f_0 + [f_s(\mathbf{v}, 0) - f_0] e^{-\nu_0 t}$$

The distribution *fo* is a **local drifting (anisotropic) Maxwellian,** or a 5-moments-approximation Maxwellian, slightly departed from the equilibrium distribution, this means that

$$|\mathbf{u}| \ll \sqrt{k_B T / m}$$

holds for any **r**.

The model is useful because its simplicity, but it gives identical relaxation times for all the moments (density, momentum, energy…) what **is not true.**

However, it can be used to introduce the method to compute the transport coefficients (as plasma conductivities) as follows:

By linearizing this Eq. assuming a small deviation *f1* :

$$f = f_0 + f_1$$

for the perturbed electron distribution. By substitution in the Kinetic equation, *f1* is solved and the electric current density and heat flux can be found.

The coefficients are used to **give a closure** for the hierarchy of fluid equatios.

Thus, up to first order, in a non-magnetized plasma, the scalar electrical conductivity comes from:

$$\frac{Ze\mathbf{E}}{m} \cdot \frac{\partial f_0}{\partial \mathbf{v}} = -\nu_c f_1 \qquad \sigma \mathbf{E} = \mathbf{j} = Ze \int \mathbf{v} f \, \mathrm{d}\mathbf{v} = Ze \int \mathbf{v} f_1 \, \mathrm{d}\mathbf{v}$$

$$\sigma = \frac{nZ^2e^2}{m\nu_c}$$

As greater the collision frequency is the plasma is less conductive.

Also, the heat flux is mainly produced by the electrons, the thermal conductivity *k* relates the heat flux and temperature gradient, so that :

$$f_0 \mathbf{v} \cdot \frac{\partial T}{\partial \mathbf{r}} \left(\frac{mv^2}{2k_B T^2} - \frac{5}{2T} \right) = -\nu_c f_1 \qquad \mathbf{q} = \int \frac{1}{2} m v^2 \mathbf{v} f \, \mathrm{d}\mathbf{v} = \frac{1}{2} m \int v^2 \mathbf{v} f_1 \, \mathrm{d}\mathbf{v}$$

$$\nu_c \mathbf{q} = \frac{5m}{4T} \int v^2 \mathbf{v}\mathbf{v} \cdot \nabla T f_0 \, \mathrm{d}\mathbf{v} - \frac{m^2}{4k_B T^2} \int v^4 \mathbf{v}\mathbf{v} \cdot \nabla T f_0 \, \mathrm{d}\mathbf{v} = -\frac{5nk_B^2 T}{2m} \nabla T$$

$$\kappa = \frac{5nk_B^2 T}{2m\nu_c}$$

note: $\mathbf{vv} = \mathbf{v} \otimes \mathbf{v}$

Non realistic collision term:
all fluxes evolve with equal frequency, a **non real description**.
For weakly ionized plasmas (charge-neutral collisions are dominant) is a good approximation, giving drift ***uE*** and diffussion coeffcients ***D*** for the fluxes:

$$\left. \begin{aligned} \mathbf{\Gamma}_i &= -D_i \nabla n_i + n_i \mu_i \mathbf{E} \\ \mathbf{\Gamma}_e &= -D_e \nabla n_e + n_e \mu_e \mathbf{E} \end{aligned} \right\}$$

The Fokker-Planck –Landau equation (FP).

For grazing (smooth) Coulomb collisions in a Debye sphere with large number of particles (mainly for fully ionized plasma) a drift-diffusion collision operator can be derived.

Similar to a Brownian motion, f evolves due to its 2 first moments, an expansion up to the f second derivative is possible for the Boltzmann collision term.

$$f(\mathbf{r}, \mathbf{v}, t) = \int \mathrm{d}(\Delta\mathbf{v}) \bigg\{ f(\mathbf{r}, \mathbf{v}, t - \Delta t)\psi(\mathbf{v}, \Delta\mathbf{v}) - \Delta\mathbf{v} \cdot \frac{\partial}{\partial \mathbf{v}}(f\psi) + \frac{1}{2}\Delta\mathbf{v}\Delta\mathbf{v} : \frac{\partial^2 (f\psi)}{\partial\mathbf{v}\partial\mathbf{v}} + \cdots \bigg\}$$

$$\left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}} = -\frac{\partial}{\partial \mathbf{v}} \cdot \left(\frac{f\langle\Delta\mathbf{v}\rangle}{\Delta t}\right) + \frac{1}{2}\frac{\partial^2}{\partial\mathbf{v}\partial\mathbf{v}} : \left(\frac{f\langle\Delta\mathbf{v}\Delta\mathbf{v}\rangle}{\Delta t}\right)$$

where : means dyadic product, and $\psi(\mathbf{v}, \Delta\mathbf{v})$ is a probability function measuring the averages of $\Delta\mathbf{v}$ and $\Delta\mathbf{v}\,\Delta\mathbf{v}$ (drift and diffusion coeffcients) accounting for the effect of Coulomb scattering.

$$\int \psi\, \mathrm{d}(\Delta\mathbf{v}) = 1 \qquad \left\{ \begin{array}{c} \langle\Delta\mathbf{v}\rangle \\ \langle\Delta\mathbf{v}\Delta\mathbf{v}\rangle \end{array} \right\} = \int \psi(\mathbf{v}, \Delta\mathbf{v}) \left\{ \begin{array}{c} \Delta\mathbf{v} \\ \Delta\mathbf{v}\Delta\mathbf{v} \end{array} \right\} \mathrm{d}(\Delta\mathbf{v})$$

Several approaches have led to the same form of the plasma Fokker-Planck equation (FP) (not only as an approximation of the Boltzmann collision term ii)

In Furth, H.P., Killeen, J. and Rosenbluth, M.N. *Phys. Fluids* ***6, 459.*** (1963).

from classical Coulomb scattering with impact parameter b_0, the drift and diffusion coeff. can be computed in terms of a parameter depending on the Coulomb Logarithm, of order 10.

$$\Gamma = \frac{z^2 e^4}{4\pi \varepsilon_0^2 m^2} \ln(\lambda_{\mathrm{D}}/b_0) \approx \frac{z^2 e^4}{4\pi \varepsilon_0^2 m^2} \ln \Lambda$$

The drift and diffusion coefficients can be computed by using the formalism of deriving them from two scalar potentials by simply derivation.

These Rosenbluth potentials depend on the distribution (unknown !!!) and the resulting collision term provides an integro-differential equation.

With these Rosenbluth potentials (for several species of atomic *number* z_s) usually defined as

$$G(\mathbf{v}) = \sum_s z_s^2 \int |\mathbf{v} - \mathbf{v}_s| \, f_s(\mathbf{v}_s) \, \mathrm{d}\mathbf{v}_s \qquad H(\mathbf{v}) = \sum_s z_s^2 \left(\frac{m + m_s}{m_s} \right) \int \frac{f_s(\mathbf{v}_s)}{|\mathbf{v} - \mathbf{v}_s|} \, \mathrm{d}\mathbf{v}_s$$

the drift vector and the diffusion second-rank tensor can be obtained and the FP equation reads:

$$\left(\frac{\partial f}{\partial t} \right)_{\mathrm{c}} = -\Gamma \frac{\partial}{\partial \mathbf{v}} \cdot \left(f \frac{\partial H}{\partial \mathbf{v}} \right) + \frac{1}{2} \Gamma \frac{\partial^2}{\partial \mathbf{v} \partial \mathbf{v}} : \left(f \frac{\partial^2 G}{\partial \mathbf{v} \partial \mathbf{v}} \right)$$

Or, in index notation (sum over repeated indexes)

$$\left(\frac{\partial f}{\partial t} \right)_c = -\Gamma \frac{\partial}{\partial v_i} \left[f \frac{\partial H}{\partial v_i} - \frac{1}{2} \frac{\partial}{\partial v_j} \left(f \frac{\partial^2 G}{\partial v_i \partial v_j} \right) \right] \equiv -\Gamma \frac{\partial}{\partial v_i} \left[D_i - \frac{\partial}{\partial v_j} D_{ij} \right] f$$

That has the form of a typical advection-diffusion equation, like the heat equation in Pyhsics.

In compact form (Fokker-Planck-Landau form) reads :

$$\left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}} = \frac{\Gamma m}{2}\frac{\partial}{\partial \mathbf{v}} \cdot \sum_s z_s^2 \int \mathrm{d}\mathbf{v}_s \frac{\partial^2 |\mathbf{v} - \mathbf{v}_s|}{\partial \mathbf{v} \partial \mathbf{v}} \cdot \left(\frac{f_s(\mathbf{v}_s)}{m}\frac{\partial f(\mathbf{v})}{\partial \mathbf{v}} - \frac{f(\mathbf{v})}{m_s}\frac{\partial f_s(\mathbf{v}_s)}{\partial \mathbf{v}_s}\right)$$

This FP collision term obeys the expected physics of the problem:

1.- The distribution *fs* **is always positive** (if it is initially positive).
2.-**System Momentum and energy are conserved** at any *t*.
3.- In absence of spatial anisotropies and forces,
the **steady state solution** is a **Maxwellian.**
4.-Typical relaxation times for momentum and energy can be estimated for each plasma species.

Braginskii fluid equations use transport coefficients derived from numerical solutions of the FP eq. See.NRL Plasma Formulary pg. 35-39

A special case: Electrons colliding with massive ions at rest:

$$\left(\frac{\partial f_e}{\partial t}\right)_c = \frac{n_e Z e^4 \ln\Lambda}{8\pi\varepsilon_0^2 m^2}\frac{\partial}{\partial \mathbf{v}}\cdot\left(\frac{v^2\mathbf{I}-\mathbf{vv}}{v^3}\cdot\frac{\partial f_e}{\partial \mathbf{v}}\right)$$

gives a value for the electron-ion collision frequency $\nu_{ei} = \dfrac{Z n_e e^4 \ln\Lambda}{4\sqrt{2}\pi\varepsilon_0^2 m^{1/2} T_e^{3/2}}$

Some text books dealing with kinetic theory in plasmas:

.-***Principles of Plasma Physics for Engineers and Scientists***, U S Inan and M Gołkowski, Cambridge University Press, 2011.

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- ***The physics of plasmas***, TJ Boyd, JJ Sanderson. Cambridge University Press, 2007.

.- ***Ionospheres: Physics, Plasma Physics, and Chemistry***,RW. Schunk, A F Nagy.Cambridge 2000

.- F. L. Hinton , ***Collisional transport in plasma*****,** in Handbook of Plasma Physics, 1983. (pdf online)

Some papers dealing with the FP eq and transport coeff. (numerical solutions with interesting theoretical introductions, **see references therein**)

F. Filbet, L. Pareschi, *A Numerical Method for the Accurate Solution of the Fokker–Planck–Landau Equation in the Nonhomogeneous Case*, Journal of Computational Physics, Volume 179, pg.1-26 (2002).

R Duclous, B Dubroca, F Filbet, V Tikhonchuk, *High order resolution of the Maxwell–Fokker–Planck–Landau model intended for ICF applications*, Journal of Computational Physics, Volume 228, pg. 5072-5100 (2009).

Ecuaciones de la física de plasmas en otros campos, un ejemplo:

PROCEEDINGS OF THE ROYAL SOCIETY A

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Boltzmann and Fokker–Planck equations modelling opinion formation in the presence of strong leaders

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We propose a mathematical model for opinion formation in a society that is built of two groups, one group of 'ordinary' people and one group of 'strong opinion leaders'. Our approach is based on an opinion formation model introduced in Toscani (Toscani 2006 *Commun. Math. Sci.* **4**, 481–496) and borrows ideas from the kinetic theory of mixtures of rarefied gases. Starting from microscopic interactions among individuals, we arrive at a macroscopic description of the opinion formation process that is characterized by a system of Fokker–Planck-type equations. We discuss the steady states of this system, extend it to incorporate emergence and decline of opinion leaders and present numerical results.

Keywords: Boltzmann equation; Fokker–Planck equation; opinion formation; sociophysics

1. Introduction

http://rspa.royalsocietypublishing.org/content/early/2009/09/02/rspa.2009.0239.full