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An Introduction to Plasma Physics and its Space Applications, Volume 2

Basic equations and applications

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Chapter 2

Elements of plasma kinetic theory

In this chapter we introduce the advanced *kinetic* description of non-equilibrium plasmas that generalizes the statistical concepts for neutral gases in equilibrium introduced in chapter 3 of volume 1. The Boltzmann equation governs the time evolution of the probability distribution function of electrons, ions and neutral atoms. The evolution introduced by collisions at the microscopic level is discussed on the basis of the simple BGK relaxation model and the Boltzmann collision integral.

The plasma kinetic theory considers the plasma as an ensemble of interacting electrically charged particles and neutral atoms or molecules. However, we will consider in the following only one kind of particle to focus on basic concepts that will be later extended to all plasma species.

The dynamical state of a particle is determined in classical mechanics by its position $\boldsymbol{r}$ and velocity $\boldsymbol{v}$ (or momentum $\boldsymbol{p}$) at any given time. Then, we can represent a system of N particles by a collection of pairs $(\boldsymbol{r}_i, \boldsymbol{v}_i)$ corresponding to the $i = 1, 2, \ldots N$ particles, equivalent to N points distributed into a six-dimensional *phase space* $(\boldsymbol{r}, \boldsymbol{v}) = (r_x, r_y, r_z, v_x, v_y, v_z)$. The velocity $\boldsymbol{v}$ and position $\boldsymbol{r}$ are independent variables that are respectively called *velocity space* and *geometry space*.

For a large number of particles it is statistically meaningful to consider the number of particles (points) dN contained within a small volume element $d^3r\ d^3v = dx\ dy\ dz\ dv_x\ dv_y\ dv_z$ in six dimensions of the phase space located at point $(\boldsymbol{r}, \boldsymbol{v})$. Equivalently, dN is also the number of particles within the volume located between $\boldsymbol{r}$ and $\boldsymbol{r} + d\boldsymbol{r}$ in the geometry space with velocities between $\boldsymbol{v}$ and $\boldsymbol{v} + d\boldsymbol{v}$. We can also write $d^3v\ d^3r = d\boldsymbol{v}\ d\boldsymbol{r}$ to simplify the mathematical notation and,

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$$dN = f(\boldsymbol{v}, \boldsymbol{r}, t)\, d^3v\, d^3r = f(\boldsymbol{v}, \boldsymbol{r}, t)\, d\boldsymbol{v}\, d\boldsymbol{r} \tag{2.1}$$

where $f(\boldsymbol{v}, \boldsymbol{r}, t)$ is the *velocity distribution function*[1]. Within an elementary volume (d^3r) of the geometrical space, the number density of particles dn is,

$$dn = \frac{dN}{dr^3} = f(\boldsymbol{v}, \boldsymbol{r}, t)\, d^3v \quad \text{also,} \quad \delta N = \frac{dN}{dr^3\, dv^3} = f(\boldsymbol{v}, \boldsymbol{r}, t)$$

Then $n(\boldsymbol{r}, t)$ is the particle density function, whereas δN represents the number of points contained within the volume $d^3r\, d^3v$ of the phase space. Therefore the function,

$$P(\boldsymbol{v}, \boldsymbol{r}, t) = \frac{\delta N}{N} = \frac{1}{N} \times \frac{dN}{dr^3\, dv^3} = \frac{1}{N} f(\boldsymbol{v}, \boldsymbol{r}, t) \tag{2.2}$$

gives the probability $P(\boldsymbol{v}, \boldsymbol{r}, t)$ of finding one particle with velocity in the range $(\boldsymbol{v}, \boldsymbol{v} + d\boldsymbol{v})$ and position between $(\boldsymbol{r}, \boldsymbol{r} + d\boldsymbol{r})$ in the geometry space. This *probability distribution function* evolves in time, and is normalized to the unity,

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(\boldsymbol{v}, \boldsymbol{r}, t)\, d\boldsymbol{v}\, d\boldsymbol{r} = 1 \text{ and equivalently,} \quad \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\boldsymbol{v}, \boldsymbol{r}, t)\, d\boldsymbol{v}\, d\boldsymbol{r} = N \tag{2.3}$$

where N is the total number of particles in the system. The integrals are extended to all velocities and positions possible to cover the entire $(\boldsymbol{r}, \boldsymbol{v})$ phase space.

The macroscopic energy of the system is finite, so the sum of the kinetic energies of all particles must be bounded. To achieve this, the probability of finding a rapidly moving particle must decrease as its speed $v = |\boldsymbol{v}|$ increases at all points $\boldsymbol{r}$ of the geometry space. Otherwise, the macroscopic energy of the system would grow without limit. Mathematically, this means that the function $f(\boldsymbol{v}, \boldsymbol{r}, t)$ tends to zero when the speed $|\boldsymbol{v}|$ becomes infinitely large at all positions of $\boldsymbol{r}$ of the geometry space.

The *stationary* $f(\boldsymbol{v}, \boldsymbol{r})$ velocity distribution function has $\partial f/\partial t = 0$. When it is independent of $\boldsymbol{r}$ as $f(\boldsymbol{v}, t)$ it is said to be *homogeneous*, and *inhomogeneous* if it is not uniform in the geometry space. The distribution is *isotropic* when $f(|\boldsymbol{v}|, \boldsymbol{r}, t)$ is independent of the direction of the velocity $\boldsymbol{v}$ and it is called *anisotropic* otherwise.

The equilibrium Maxwell–Boltzmann distribution, equation (3.5) of volume 1, is a *stationary*, *homogeneous* and *isotropic* probability distribution function. Its properties reflect the physical characteristics of the thermodynamic equilibrium state at microscopic level; the molecules are equally distributed in space and their motions have no preferred direction since all are equally likely.

As in section 3.3 in volume 1, we can introduce the macroscopic physical magnitudes as averages of the distribution function $f(\boldsymbol{v}, \boldsymbol{r}, t)$ over the phase space $(\boldsymbol{r}, \boldsymbol{v})$. Since $\delta N = f(\boldsymbol{v}, \boldsymbol{r}, t)\, d\boldsymbol{v}\, d\boldsymbol{r}$ gives the number of points inside the elementary

[1] It will be called also *distribution function* in the following for short.

volume ($d^3r\ d^3v$) and $n(\boldsymbol{r},\ t)$ is the number density of particles within d^3r, integration gives

$$n(\boldsymbol{r},\ t) = \int_{-\infty}^{+\infty} f(\boldsymbol{v},\ \boldsymbol{r},\ t)\ d\boldsymbol{v}\ \text{ and also, }\ N = \int_{-\infty}^{+\infty} n(\boldsymbol{r},\ t)\ d\boldsymbol{r} \tag{2.4}$$

We can also introduce the average or *macroscopic flow velocity*,

$$\boldsymbol{u}(\boldsymbol{r},\ t) = \frac{\int_{-\infty}^{+\infty} \boldsymbol{v}\, f(\boldsymbol{v},\ \boldsymbol{r},\ t)\ d\boldsymbol{v}}{\int_{-\infty}^{+\infty} f(\boldsymbol{v},\ \boldsymbol{r},\ t)\ d\boldsymbol{v}} = \frac{1}{n_\alpha(\boldsymbol{r},\ t)} \int_{-\infty}^{+\infty} \boldsymbol{v}\, f(\boldsymbol{v},\ \boldsymbol{r},\ t)\ d\boldsymbol{v} \tag{2.5}$$

which gives the *velocity field* $\boldsymbol{u}(\boldsymbol{r},\ t)$ and the *flux of particles* as,

$$\boldsymbol{\Gamma}(\boldsymbol{r},\ t) = \int_{-\infty}^{+\infty} \boldsymbol{v} f(\boldsymbol{v},\ \boldsymbol{r},\ t) d\boldsymbol{v} \tag{2.6}$$

These averages generalize concepts previously introduced in volume 1 section 3.2 for the Maxwellian distribution equations (3.5) or (3.7) of a neutral gas.

The determination of the distribution function is a central problem of kinetic theory since macroscopic physical properties are calculated as statistical averages. The evolution of $f(\boldsymbol{v},\ \boldsymbol{r},\ t)$ is governed by the *Boltzmann* or *kinetic equation* which gives the temporal and spatial evolution of the distribution function. However, the *probability distribution function* can be used instead and $dP = P(\boldsymbol{v},\ \boldsymbol{r},\ t)\ d\boldsymbol{v}\ d\boldsymbol{r}$ gives the probability of finding one particle within the elementary volume ($d^3r\ d^3v$) of the phase space.

2.1 The Boltzmann equation

We are now in a position to generalize the concepts introduced in the previous section for a system composed of several different kinds of particles. In the kinetic description the N_α plasma particles, where $\alpha = e,\ i,\ a$ labels electrons, ions and neutral atoms, these are considered as points with mass m_α and electric charge q_α (with $q_a = 0$) distributed into the *phase space* $(\boldsymbol{r},\ \boldsymbol{v})$. Each plasma species has its own velocity distribution function $f_\alpha(\boldsymbol{v},\ \boldsymbol{r},\ t)$ that—as we shall see—is coupled by the elementary processes (collisions) between them at the microscopic level.

We will introduce here the Boltzmann equation by means of a somewhat simple and intuitive approach, starting from the definition (2.2) of the probability distribution function. The rigorous formulation in connection to the positions and velocities of point-like charged plasma particles and the derivation of the Boltzmann (or kinetic) equation is outlined[2] in appendix B.

[2] A more rigorous derivation of plasma kinetic equations (B.15) or (B.16) is in appendix B connecting the point-like distribution of particles in phase space with the probability distribution (2.2). The function $f_\alpha(\boldsymbol{r},\ \boldsymbol{v},\ t)$ results from the average over an intermediate length scale L_c much longer than the interparticle distance $d_o = n_o^{-1/3} \ll L_c \lesssim \lambda_D$ but below the Debye length and velocities $v_c = L_c/T_\alpha \lesssim \lambda_D \times f_{p,\alpha}$ where $f_{p,\alpha} = 1/T_\alpha$ is the plasma frequency which introduces the faster time of response of the plasma.

The distribution function $f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)$ depends on seven independent variables[3] and using equation (A.3) its total time derivative is,

$$\frac{Df_\alpha}{Dt} = \frac{\partial f_\alpha}{\partial t} + \sum_i \left(\frac{\partial f_\alpha}{\partial x_i}\frac{dx_i}{dt}\right) + \sum_i \left(\frac{\partial f_\alpha}{\partial v_i}\frac{dv_i}{dt}\right)$$

that may be cast (equation A.4) as,

$$\frac{Df_\alpha}{Dt} = \frac{\partial f_\alpha}{\partial t} + \boldsymbol{v} \cdot \nabla_{\mathrm{r}} f_\alpha + \boldsymbol{a} \cdot \nabla_{\mathrm{v}} f_\alpha$$

In this equation,

$$\nabla_{\mathrm{r}} \equiv \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right) \text{ and, } \nabla_{\mathrm{v}} \equiv \left(\frac{\partial}{\partial v_x}\boldsymbol{i} + \frac{\partial}{\partial v_y}\boldsymbol{j} + \frac{\partial}{\partial v_z}\boldsymbol{k}\right)$$

respectively, are the nabla operators, ∇_{r} in the geometry space and ∇_{v} in the velocity space where $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ are constant unit vectors. In order to derive the Boltzmann equation for the distribution $f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)$ we recast the previous expression introducing the acceleration $\boldsymbol{a}$ or force by unit of mass as,

$$\boldsymbol{a} = \frac{\boldsymbol{F}}{m_\alpha} = \boldsymbol{f}_{g\alpha} + \frac{q_\alpha}{m_\alpha}(\boldsymbol{E} + \boldsymbol{v}_\alpha \times \boldsymbol{B})$$

Here $\boldsymbol{f}_{g\alpha}$ represents other non-electromagnetic forces acting on the α particles[4] with electric charge q_α. The local electromagnetic fields $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ are spatially averaged[5] over distances $L \lesssim \lambda_D$ to preserve the plasma collective effects. We can write the time evolution equation,

$$\frac{Df_\alpha}{Dt} = \frac{\partial f_\alpha}{\partial t} + \boldsymbol{v} \cdot \nabla_{\mathrm{r}} f_\alpha + \left[\boldsymbol{f}_{g\alpha} + \frac{q_\alpha}{m_\alpha}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})\right] \cdot \nabla_{\mathrm{v}} f_\alpha \tag{2.7}$$

When the non-equilibrium velocity distribution function $f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)$ eventually remains unaltered in time we have $Df_\alpha/Dt = 0$. In this case equation (2.7) is denominated the *Vlasov equation* or *collisionless Boltzmann equation* and the velocity distribution function is also a constant of motion in the dynamical evolution of the system.

However, the plasma particles exchange energy and momentum in collisional processes which produce changes in the velocity distribution function of ions, electrons and neutral atoms. Collisions alter number of particles (points) initially located in the phase space volume $(d^3r\, d^3v)$ centered at $(\boldsymbol{r}, \boldsymbol{v})$ because their velocities and/or positions can change. Then, particles move to other points of the phase space

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[3] The derivation of this scalar function DP_α/Dt and the connection between Lagrangian and Eulerian velocities are discussed in appendix A.

[4] The Boltzmann equation also applies to neutral gas particles setting the electric charge $q_\alpha = 0$, as they are not affected for the electromagnetic fields but are coupled by collisions with the electron and ion distribution functions.

[5] In appendix B it is discussed that these electromagnetic fields acting over the individual charge q_α result from a *spatial average* over typical lengths $L \lesssim \lambda_D$ of effective electric charge shielding.

after colliding because their initial velocities and positions are altered on a time scale governed by collisions. In the general case, molecular encounters alter the velocity distribution function $Df_\alpha/Dt \neq 0$ and we can write,

$$\frac{Df_\alpha}{Dt} = C(f_\alpha) \text{ where, } C(f_\alpha) = \left(\frac{\delta f_\alpha}{\delta t}\right)_{\text{col}}$$

represents the change in the velocity distribution f_α originated by collisional processes with the other plasma species. The rate of change also involves their distributions and,

$$C(f_\alpha) = \sum_\beta C_{\alpha,\beta}(f_\alpha, f_\beta) \tag{2.8}$$

where $C_{\alpha,\beta}(f_\alpha, f_\beta)$ represents the change in f_α by collisions of α particles with all other β species. At this point we do not have an explicit expression available for $C(f_\alpha)$ coupling the probability distribution of two plasma species, that will depend on the collisional cross sections, particle densities, etc, so that we will leave this point for later.

Finally, the *Boltzmann equation* for the α-particle distribution function is,

$$\frac{\partial f_\alpha}{\partial t} + \boldsymbol{v} \cdot \nabla_{\text{r}} f_\alpha + \left[\boldsymbol{f}_{g\alpha} + \frac{q_\alpha}{m_\alpha}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})\right] \cdot \nabla_{\text{v}} f_\alpha = C(f_\alpha) \tag{2.9}$$

where the *collision term* $C(f_\alpha)$ is still undefined and accounts for the dynamical evolution introduced by the encounters at the atomic and molecular level, averaged over an effective electric field shielding length[6] λ_D.

Equation (2.9) can be seen as a continuity equation for the velocity distribution function in the phase space $(\boldsymbol{r}, \boldsymbol{v})$. First, we use $\boldsymbol{v} \cdot \nabla_{\text{r}} f_\alpha = \nabla_{\text{r}} \cdot (\boldsymbol{v} f_\alpha)$ and also

$$\boldsymbol{a} \cdot \nabla_{\text{v}} f_\alpha = \left[\boldsymbol{f}_{g\alpha} + \frac{q_\alpha}{m_\alpha}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})\right] \cdot \nabla_{\text{v}} f_\alpha = \nabla_{\text{v}} \cdot (\boldsymbol{a} f_\alpha)$$

The electromagnetic fields $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ as well as $\nabla_{\text{v}} \cdot \boldsymbol{f}_{g\alpha} = 0$ are independent of $\boldsymbol{v}$, so we can also make use of,

$$\nabla_{\mathbf{v}} \cdot (\boldsymbol{v} \wedge \boldsymbol{B}) = \boldsymbol{B} \cdot \cancelto{0}{(\nabla_{\mathbf{v}} \wedge \boldsymbol{v})} + \boldsymbol{v} \cdot (\nabla_{\mathbf{v}} \wedge \boldsymbol{B})$$

Therefore, the *Boltzmann equation* (2.9) is equivalent to,

$$\frac{\partial f_\alpha}{\partial t} + \nabla_{\mathbf{r}} \cdot (\boldsymbol{v} f_\alpha) + \nabla_{\text{v}} \cdot (\boldsymbol{a} f_\alpha) = C(f_\alpha) \tag{2.10}$$

where $C(f_\alpha)$ is the sum (2.8) of all changes experienced by the distribution function for the α species by collisions. Equation (2.10) has the implicit assumption that the

[6] Strictly speaking, the Debye length introduced in section 4.5 of volume 1 is only valid for Maxwellian electrons and ions. This is not the general case, so λ_D can be considered here as an *effective electric shielding* scale length.

temporal scales of all collisional processes considered in $C(f_\alpha)$ are much faster than the time evolution of the distribution function.

Finally, introducing a generalized coordinate $\boldsymbol{R} = (\boldsymbol{r}, \boldsymbol{v})$ in the phase space and its time derivative,

$$\boldsymbol{V} = (\boldsymbol{v}, \boldsymbol{a}) = \left(\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}, \boldsymbol{a} = \frac{d\boldsymbol{v}}{dt}\right)$$

Expression (2.10) can be cast into a *generalized continuity equation* for the distribution f_α in the phase space as,

$$\frac{\partial f_\alpha}{\partial t} + \nabla_{\boldsymbol{R}} \cdot (\boldsymbol{V} f_\alpha) = C(f_\alpha) \tag{2.11}$$

Then, $C(f_\alpha)$ can be considered as a the source/sink term and the vector $\boldsymbol{q} = f_\alpha\ \boldsymbol{V}$ is the flux of $f_\alpha(\boldsymbol{R}, t) = f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)$. The collision operator $C(f_\alpha) = (\delta f_\alpha / \delta t)_{\mathrm{col}}$ accounts for points (particles) added and/or removed from the volume $(d^3r\ d^3v)$ of the phase space.

The plasma elementary processes introduced in chapters 5 and 6 of volume 1 are responsible for the term $C_s(f_\alpha)$ in the Boltzmann equation (2.9) or (2.9) which introduces the collisional changes in the probability density during the evolution of the system. In *collisionless plasmas* $C(f_\alpha) = 0$ and the probability density $f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)$ is a conserved quantity in the dynamic evolution of the system.

The Boltzmann equations (2.9) or (2.9) needs to be complemented with the Maxwell equations for the electromagnetic fields $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ in the plasma. These fields result from the coupling of plasma charged particles and the externally applied or self-generated fields. The velocity distribution function $f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)$ can be used to calculate the average plasma density of charged particles ρ_c and the transported current density $\boldsymbol{J}_c$ as,

$$\rho_c = \sum_\alpha \rho_{c\,\alpha} = \sum_\alpha q_\alpha\, n_\alpha(\boldsymbol{r}, t) = \sum_\alpha q_\alpha \int_{-\infty}^{+\infty} f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)\, d\boldsymbol{v} \tag{2.12}$$

and,

$$\boldsymbol{J}_c = \sum_\alpha \boldsymbol{J}_{e\alpha} = \sum_\alpha q_\alpha \int_{-\infty}^{+\infty} \boldsymbol{v}\, f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)\, d\boldsymbol{v} \tag{2.13}$$

that should be introduced in the Maxwell equations to self-consistently calculate the electromagnetic fields $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$.

From a formal point of view, and lacking an explicit formulation for the collisional term $C_s(f_\alpha)$, the kinetic approach to plasma physics may be understood as the closure of the Maxwell equations for $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ using Boltzmann equation (2.10).

When the distribution functions $f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)$ can be obtained, the physical macroscopic magnitudes, such as the local particle density $n_\alpha(\boldsymbol{r}, t)$ in equation (2.4) or the velocity field $\boldsymbol{u}_\alpha(\boldsymbol{r}, t)$ in equation (2.5) for ions, electrons and neutral atoms can be calculated. This approach was previously used in volume 1 sections 3.3 and 3.4 using the equilibrium Maxwell–Boltzmann distributions of equations (3.5) or (3.7).

However, to obtain explicit solutions of the Boltzmann equation is difficult. The generalization of this kinetic scheme to non-equilibrium plasmas is by far much more complex than for a neutral gas in equilibrium. Strictly speaking, we need to model the collision term $C_s(f_\alpha)$ accounting for the specific characteristics of collisional processes discussed in chapter 6 of volume 1.

The limited *relaxation model* introduced in the next section is a simple approach to introduce the effect of short-range collisions and next we will discuss the more involved *Boltzmann collision integral* valid for short-range binary collisions in low pressure gases and weakly ionized plasmas. The long-range Coulomb collisions involve the simultaneous interaction of many charged particles and are described by the more involved *Fokker–Planck* or *Landau collision operators*.

2.2 Relaxation model for molecular collisions

The formulation of specific models for the collision term $C_s(f_\alpha)$ we left aside in the previous section is a difficult task since the mathematical description of specific elementary processes are involved. This term in equations (2.9) or (2.10) accounts for the changes in time produced by molecular encounters in the velocity distribution function $f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)$. We will discuss the simple formulation of the *relaxation* or *Krook model*[7] which comes from the neutral gas particle collision term.

In the first place, the existence of a *local and stationary* distribution $f_\alpha^{eq}(\boldsymbol{r}, \boldsymbol{v})$ close to the local Maxwellian distribution (4.3) introduced in section 4.2 of volume 1 is assumed. The initial ($t = 0$) non-equilibrium distribution $f_{\alpha,o}(\boldsymbol{v}, \boldsymbol{r}, 0)$ evolves in time to f_α^{eq} at a rate,

$$\frac{\partial f_\alpha}{\partial t} = -\frac{(f_\alpha - f_\alpha^{eq})}{\tau_r} < 0$$

where $f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)$ is the distribution function for $t > 0$ and τ_r is a *characteristic relaxation time*. We obtain a differential equation for f_α as,

$$\frac{\partial f_\alpha}{\partial t} + \frac{f_\alpha}{\tau_r} = \frac{f_\alpha^{eq}}{\tau_r}$$

and the right term f_α^{eq}/τ_r is time-independent. Its complete solution is,

$$f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t) = f_\alpha^{eq}(\boldsymbol{v}, \boldsymbol{r}) + C\, e^{-t/\tau_r}$$

and $C = (f_{\alpha,o} - f_\alpha^{eq})$ can be determined using the non-equilibrium distribution $f_{\alpha,o}(\boldsymbol{v}, \boldsymbol{r}, 0)$ for the initial instant giving,

$$f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t) = f_\alpha^{eq}(\boldsymbol{v}, \boldsymbol{r}) + (f_{\alpha,o} - f_\alpha^{eq})\, e^{-t/\tau_r}$$

[7] Proposed by Bathnagar, Groos and Krook in reference [1], it is also called the BGK model.

The departure $(f_{\alpha,o} - f_\alpha^{eq})$ from the equilibrium distribution function decreases in time at a rate governed by the relaxation time τ_r and is also *spatially homogeneous* since it is independent of $\boldsymbol{r}$. Equivalently, the time-dependent amplitude $(f_{\alpha,o} - f_\alpha^{eq})$ of fluctuations from the equilibrium distribution function $f_\alpha^{eq}(\boldsymbol{r}, \boldsymbol{v})$ exponentially decays in time for small departures of the equilibrium state.

This simple *relaxation model* is practical and τ_c can be directly related with the dominant collision cross sections. In weakly ionized plasmas governed by short-range collisions of charged particles with neutral atoms we can introduce the relaxation times $\tau_{ea} \sim 1/\nu_{ea}$ and $\tau_{ia} \sim 1/\nu_{ia}$ for electron and ions. The rate of change of the ion and electron velocity distribution functions depend on the collision frequencies,

$$\nu_{ea} \sim n_a\, n_e\, \sigma_{ea}\, \bar{v}_i \text{ and also, } \nu_{ia} \sim n_a\, n_i\, \sigma_{ia}\, \bar{v}_i$$

introduced in section 3.4 of volume 1 that can be directly incorporated into this BGK model. Additionally, high neutral atom densities (proportional to the gas pressure p_a) reduce the relaxation times τ_r that govern the amplitude of fluctuations from the equilibrium plasma state. The short relaxation times facilitate a fast *collisional energy thermalization* among plasma species, as discussed in section 4.3 of volume 1.

As we shall see in chapter 3, this simple BGK model, can be employed to formulate the macroscopic momentum and energy exchange (close equilibrium state) between the electron and ion fluids in the hydrodynamic description of weakly ionized plasmas. However, except for systems where a specific collisional process dominates, this model is rather limited. Using a single time rate τ_r (equivalently, one collision cross section) the BGK model is a crude approximation that oversimplifies the relaxation phenomena of actual systems where multiple elementary processes can take place at the microscopic level, governed by different collision cross sections, etc.

2.3 The Boltzmann collision integral

The *Boltzmann collision integral* is a more refined approximation for the collision term $C_s(f_\alpha)$ in equation (2.10) than the BGK relaxation model but—as we shall see—it is still limited, since it is also based on simplifying assumptions. The Boltzmann collision integral model is applicable to sort-range binary collisions in dilute gases and plasmas where molecular or atomic forces decay much faster than the inter-particle distance. This situation takes place in weakly ionized plasmas and gases at low pressures. The spatial changes of velocity distributions $f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)$ are smooth and its temporal variations slow compared with characteristic collision time. That is, the model applies when the temporal scales of collisional processes faster than the evolution in time of the system. After its derivation we will examine with more detail the underlying physical approximations.

2.3.1 Qualitative derivation

In the following, we will consider a fixed *phase space* volume $d^3r\, d^3v_\alpha = d\boldsymbol{r}\, d\boldsymbol{v}_\alpha$ located at point $(\boldsymbol{r}, \boldsymbol{v}_\alpha)$ and the *geometry volume* $d\boldsymbol{r} = d^3r$ at point $\boldsymbol{r}$ will remain fixed and is occupied by target α-particles. Collisions with the incoming β-particles change the number of targets within the elementary volume $d\boldsymbol{v}_\alpha = d^3v_\alpha$ placed about $\boldsymbol{v}_\alpha$ in the *velocity space*.

The change in number of α-particles within the elementary phase space volume $(d^3r\ d^3v_\alpha)$ during the time interval δt can be expressed (see equation 2.1) as,

$$\delta N_\alpha = \delta N_\alpha^{\text{in}} - \delta N_\alpha^{\text{out}} = \left(\frac{\delta f_\alpha}{\delta t}\right)_{\alpha\beta} (d^3r \quad d^3v_\alpha) \quad \delta t$$

Here δN_α is the difference between the $\delta N_\alpha^{\text{in}}$ number of α-particles that come *into* the velocity volume interval d^3v_α located at point $\boldsymbol{v}_\alpha$ *after* experiencing one collision event. The number $\delta N_\alpha^{\text{out}}$ are those initially within d^3v_α that are pulled out when colliding. Additionally, $(\delta f_\alpha/\delta t)_{\text{col}}$ represents one specific binary[8] (two-particle) collision term $C_{\alpha\beta}(f_\alpha, f_\beta)$ in the sum C_α of equation (2.10).

The term $\delta N_\alpha^{\text{out}}$ accounts for *direct collisions* that pull out the α-particle initially located within $(d^3r\ d^3v_\alpha)$ from the velocity range $(\boldsymbol{v}_\alpha, \boldsymbol{v}_\alpha + d\boldsymbol{v}_\alpha)$ or equivalently, from the volume d^3v_α located at point $\boldsymbol{v}_\alpha$. The *inverse collisions* contribute to $\delta N_\alpha^{\text{in}}$ since they bring inside the volume d^3v_α placed at $\boldsymbol{v}_\alpha$ the α-particles within d^3r others located outside, with initial velocities between $\boldsymbol{v}'_\alpha$ and $\boldsymbol{v}'_\alpha + d\boldsymbol{v}'_\alpha$ before the encounter.

The number of incoming β-particles scattered by *one* α-target by *direct collisions* using equation (5.4) of volume 1 is,

$$\delta \dot{Q}_{\text{dir}} \times \delta t = [\, \sigma_{\text{dir}}(g, \chi)\ d\Omega\,] \times \Gamma_\beta \times \delta t$$

where $\Gamma_\beta = g\, f_\beta(\boldsymbol{v}_\beta, \boldsymbol{r}, t)\ d^3v_\beta$ is the flux (2.6) of β-particles, $\boldsymbol{g} = \boldsymbol{v}_\beta - \boldsymbol{v}_\alpha$ is the relative velocity, the collision cross section is $\sigma_{\text{dir}}(g, \chi)$ and $d\Omega$ the solid angle of figure 5.2 in volume 1. If we multiply by the number of α-particle targets within the d^3r geometry space volume,

$$\left[f_\alpha(\boldsymbol{v}_\alpha, \boldsymbol{r}, t)\ d^3v_\alpha\ d^3r \right] \times [\, \sigma_{\text{dir}}(g, \chi)\ d\Omega\,] \times g \times \left[f_\beta(\boldsymbol{v}_\beta, \boldsymbol{r}, t)\ d^3v_\beta \right] \times \delta t$$

we obtain the total number of encounters. Integrating for all possible velocities $\boldsymbol{v}_\beta$ and scattering angles $d\Omega$ we derive the contribution of direct collisions,

$$\delta N_\alpha^{\text{out}} = (\delta t\ d^3r) \times \int_{-\infty}^{+\infty} \int_\Omega g\ \sigma_{\text{dir}}(g, \chi)\ d\Omega\, f_\alpha\, f_\beta \quad d^3v_\alpha\ d^3v_\beta \tag{2.14}$$

where we have introduced the notation $f_\alpha = f_\alpha(\boldsymbol{v}_\alpha, \boldsymbol{r}, t)$ and $f_\beta = f_\beta(\boldsymbol{v}_\beta, \boldsymbol{r}, t)$ for short.

A similar argumentation is used for the *inverse* collisions where again the β-particles scattered by one α-target is,

$$\delta \dot{Q}_{\text{inv}} \times \delta t = [\, \sigma_{\text{inv}}(g', \chi')\ d\Omega'\,] \times \Gamma'_\beta \times \delta t$$

[8] This specific approximation is crucial as collisions involving three or more particles are beyond this approach, valid only for low pressure (or *dilute*) gases.

where $\Gamma'_\beta = g' f_\beta(\boldsymbol{v}'_\beta, \boldsymbol{r}, t)\, d^3v'_\beta$ is the flux of incoming particles and $\boldsymbol{g}' = \boldsymbol{v}'_\beta - \boldsymbol{v}'_\alpha$ their relative velocity. Multiplying by the α-targets inside d^3r we have,

$$\left[f_\alpha(\boldsymbol{v}'_\alpha, \boldsymbol{r}, t)\, d^3v'_\alpha\, d^3r \right] \times [\, \sigma_{\rm inv}(g', \chi')\, d\Omega' \,] \times g' \times \left[f_\beta(\boldsymbol{v}'_\beta, \boldsymbol{r}, t)\, d^3v'_\beta \right] \times \delta t$$

Integrating over all velocities $\boldsymbol{v}'_\beta$ and scattering angles $d\Omega'$ we obtain the contribution of direct collisions,

$$\delta N_\alpha^{\rm in} = (\delta t\ d^3r) \times \int_{-\infty}^{+\infty} \int_{\Omega'} g'\, \sigma_{\rm inv}(g', \chi')\, f'_\alpha f'_\beta\ d\Omega'\ \ d^3v'_\alpha\, d^3v'_\beta \tag{2.15}$$

where again $f'_\alpha = f'_\alpha(\boldsymbol{v}'_\alpha, \boldsymbol{r}, t)$ and $f'_\beta = f'_\beta(\boldsymbol{v}_\beta, \boldsymbol{r}, t)$. Finally, using equations (2.14) and (2.15) the collisional term can be calculated as,

$$C_{\alpha\beta}(f_\alpha, f_\beta) = \frac{\delta N_\alpha^{\rm in} - \delta N_\alpha^{\rm out}}{d^3v_\alpha\ d^3r\ \delta t} \tag{2.16}$$

that can be understood as a balance equation accounting for the number of incoming and outgoing particles from the elementary volume element of the phase space ($d^3r\ d^3v_\alpha$) caused by collisions. It is important to note that a specific model for the cross sections $\sigma_{\rm dir}(g, \chi)$ and $\sigma_{\rm dir}(g, \chi)$ is still required in equations (2.14) and (2.15). That is, the specific physical details of inter-particle interactions at the atomic and molecular level.

The above expression for $C_{\alpha\beta}(f_\alpha, f_\beta)$ takes a simple expression for *short range elastic collisions* where $g = g'$ since the vectors $\boldsymbol{g}$ and $\boldsymbol{g}'$ only differ in direction (see section 5.3 of volume 1) in this case. Additionally, $\sigma_{\rm dir}(g, \chi)\, d\Omega = \sigma_{\rm inv}(g', \chi')\, d\Omega'$ because the cross section is equal for forward and reverse collisions and assuming interparticle forces to have central symmetry is also invariant to an inversion of the coordinates[9]. Furthermore, using the center of mass $\boldsymbol{r}_{CM}$ we have,

$$d\boldsymbol{v}_\alpha\, d\boldsymbol{v}_\beta = d\boldsymbol{g}\, d\boldsymbol{r}_{cm} = d\boldsymbol{g}'\, d\boldsymbol{r}'_{cm} = d\boldsymbol{v}'_\alpha\, d\boldsymbol{v}'_\beta$$

With these simplifications we finally have,

$$\left(\frac{\delta f_\alpha}{\delta t}\right)_{\rm col} = \int_{-\infty}^{+\infty} \int_{\Omega} g\ \sigma_{\alpha\beta}(g, \chi)\ (f'_\alpha f'_\beta - f_\alpha f_\beta)\ d\Omega\ d\boldsymbol{v}_\beta \tag{2.17}$$

It is important to note that *Boltzmann collision integral* couples the velocity distribution functions of two particle species through the collisional cross section of a specific elementary process. Then equation (2.10) is transformed into an integro-differential equation when collisions are taken into account as,

[9] The *direct collision* can be viewed replacing in figure 5.2 of volume 1 the target b by α and the incoming particle α is β. In the *reverse* encounter the incoming particle moves along the opposite direction that a point follows in the figure and symmetry gives equal solid angles for both processes.

$$\frac{\partial f_\alpha}{\partial t} + \nabla_{\mathrm{r}} \cdot (v_\alpha f_\alpha) + \nabla_{\mathrm{v}} \cdot (\boldsymbol{a} f_\alpha) = \int_{-\infty}^{+\infty} \int_\Omega g \, \sigma_{\alpha\beta}(g, \chi) \times \left(f'_\alpha f'_\beta - f_\alpha f_\beta\right) d\Omega \, dv_\beta \tag{2.18}$$

Each plasma species $\alpha = e$, i, a is governed by a Boltzmann equation (2.10) with its specific collision integral (2.17) for each elementary process. The sum of all contributions gives,

$$C(f_\alpha) = \sum_{\beta \neq \alpha} C_{\alpha,\beta}(f_\alpha, f_\beta)$$

and therefore the velocity distribution functions of plasma species are coupled through collision integrals, that include the cross sections of collisional processes.

2.3.2 Approximations

The previous derivation of the Boltzmann equation (2.18) relies on the following relevant physical assumptions and/or approximations:

1. Only short-range *binary* collisions are taken into account and collective effects are ignored. This assumption is justified for low pressure gases or partially ionized low pressure plasmas where the collisions with neutral particles are dominant. This is not the case when the contribution of long-range Coulomb collisions are important.
2. The colliding particles are considered an isolated system in the above derivation. The effect of external forces on cross sections or in the two-particle collision parameters are neglected. The relative speed $g = g'$ is constant only in the absence of external forces and takes a cumbersome expression otherwise.
3. Within the phase space volume ($d^3r \; d^3v_\alpha$) the velocity distribution functions are uniform and constant in time. The collision time interval δt is much shorter than the variation in time of the velocity distribution functions of particles and d^3r much smaller than the spatial variations of the distribution functions. In other words, the temporal scale of elementary processes are faster than the time evolution of the system.
4. The velocities and positions of particles before the encounter are independent (uncorrelated). The probability that an α-particle collides with the β-particle can be considered to be proportional to the product $f_\alpha \times f_\beta$ and correlations are neglected. This assumption is known as the *molecular chaos hypothesis*.

2.3.3 The Maxwell–Boltzmann distribution

The Boltzmann equation is valid for low pressure gases and plasmas where long-range Coulomb collisions can be neglected. This is usually the case of plasmas in space and technological applications where ionization degree is low and gas pressures are moderate. In these conditions the equilibrium Maxwell–Boltzmann velocity distribution (3.5) or (3.7) is recovered from the collision integral (2.17).

In the limit of thermodynamic equilibrium the velocity distribution function $f_\alpha(\boldsymbol{v}, \boldsymbol{r})$ is stationary and a sufficient condition is to satisfy $(\delta f_\alpha/\delta t)_{\rm col} = 0$ and then,

$$\int_{v_\beta} \int_\Omega g\ \sigma_{\alpha\beta}(g, \chi)\ (f'_\alpha f'_\beta - f_\alpha f_\beta)\ d\Omega\ d\boldsymbol{v}_\beta = 0 \text{ then}, f'_\alpha f'_\beta - f_\alpha f_\beta = 0 \tag{2.19}$$

This functional equation has a *unique* solution that can be justified as follows. In the equilibrium state the distributions $f_\alpha(|\boldsymbol{v}_\alpha|) = f_\alpha(v_\alpha^2)$ are *uniform* in space (independent of $\boldsymbol{r}$) and *isotropic* $f_\alpha(|\boldsymbol{v}_\alpha|) = f_\alpha(v_\alpha^2)$ so we can write,

$$\ln f_\alpha(v_\alpha^2) + \ln f_\beta(v_\beta^2) = \ln f_\alpha(v'^2_\alpha) + \ln f_\beta(v'^2_\beta) \tag{2.20}$$

The unique solution possible to satisfy equations (2.19) and (2.20) and the conservation of energy in the collisions is,

$$f_\alpha(v_\alpha^2) = A \exp(B\ v_\alpha^2)$$

and the Maxwell–Boltzmann distribution (3.5) can be derived as in section 3.2 of volume 1 using the normalization condition for the A constant.

2.4 Commentaries and further reading

The books [2, 3] and [4] discuss in more detail the fundamentals of the plasma kinetic theory introduced in this chapter. The book [5] is an advanced text as is also the classical and comprehensive book [6]. Here we are mainly interested in partially ionized plasmas at low pressure where short-range collisions are dominant. The Boltzmann integral is in this case a valid physical description, and in chapter 21, sections 4.2 and 4.3 of reference [2] is introduced its application to ion and electron transport weakly ionized gases. This specific model is of interest in the physical description of low pressure electric discharges.

The Boltzmann equation and collision integral have played an important role connecting the deterministic laws of classical mechanics with the irreversible nature of non-equilibrium physical processes. The literature on the Boltzmann equation is huge and its mathematical properties are still a subject of research [7]. This chapter is only a brief overview to a challenging field with implications far beyond physics. Boltzmann's ideas have found applications in fields as diverse as econometric models [8], stock market trade [9], economic wealth [10] or social sciences [11].

The long-range Coulomb collisions require more involved models such as the Fokker–Planck and Landau collision operators. So far we are concerned with weakly ionized plasmas so they are outside the scope of the present work. These collision operators are discussed in chapter 21, sections 5.2 and 5.3 of the book [2] and also in chapter 3 of [12].

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