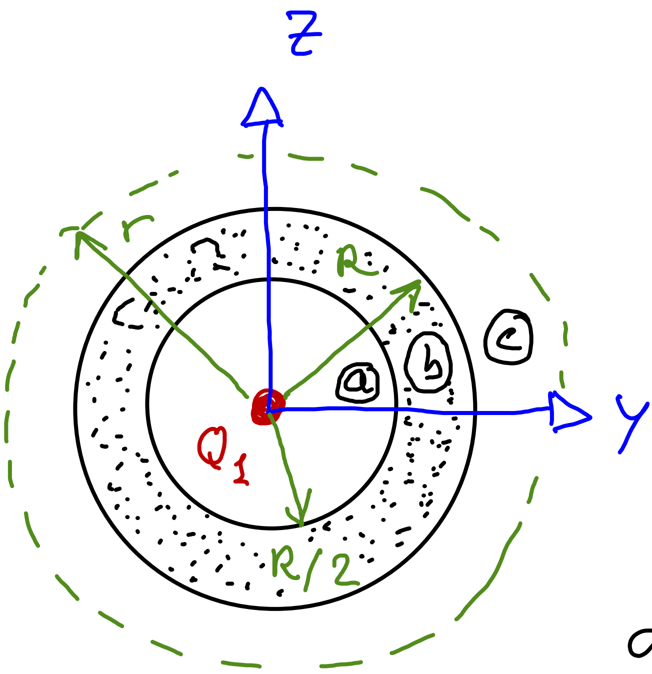

Prob: 3.11

El sistema está compuesto de dos elementos

- Carga Q_1 (zona "a")
- La distribución de carga (zona "b")

Primero calculamos la energía de la distribución de carga sin Q_1 empleando:

$$U_e = \frac{1}{2}\int_{\text{corteza}} \phi(\bar r')\,\rho(\bar r')\,dV'$$

donde la integral está extendida sobre la corteza donde $\rho(\bar r') \neq 0$.

Para calcula el potencial $\phi(\bar r')$ calculamos el campo eléctrico empleando el teorema de Gauss. Hay tres zonas distintas en el esquema de la figura

ⓐ Para $r < \frac{R}{2}$ $\quad \int_{S_a} \vec E_a \cdot d\vec S = \frac{Q_{TOT}}{\epsilon_0} = 0 \qquad \vec E_a = 0$

Como sólo se considera la corteza no hay carga

ⓑ Para $\frac{R}{2} < r < R$ el campo tiene simetría esférica y entonces $\vec E_b = \frac{\bar r}{r} = \vec e_r$

Carga encerrada dentro de la sup. esférica

$$\int_{\text{sup}} \vec E_b \cdot d\vec S = |\bar E_b|\,(4\pi r^2) = \frac{Q_{TOT}}{\epsilon_0} = \frac{1}{\epsilon_0}\int_{r > R/2} \rho_0\, dV \qquad \frac{R}{2} < r < R$$

$$|\bar E_b|\,(4\pi r^2) = \frac{1}{\epsilon_0}\int_{R/2}^{r} \rho_0 \underbrace{(4\pi r^2)\,dr}_{dV} = \frac{4\pi\rho_0}{\epsilon_0}\int_{R/2}^{r} r^2\,dr$$

$$|\bar E_b|\,(4\pi r^2) = \frac{4\pi\rho_0}{\epsilon_0}\left[\frac{r^3}{3}\right]_{R/2}^{r} = \frac{4\pi\rho_0}{\epsilon_0}\left[\frac{r^3}{3} - \frac{R^3}{24}\right]$$

$$|\vec{E}_b|\,(\cancel{4\pi}r^2) = \frac{\cancel{4\pi}\rho_0}{3\epsilon_0}\left[r^3 - \frac{R^3}{8}\right]$$

$$\vec{E}_b = \frac{\rho_0}{3\epsilon_0}\left[r - \frac{R^3}{8r^2}\right]\vec{e}_r \qquad \frac{R}{2} < r < R$$

Ⓒ Para $r \geq R$ contribuye al campo $\vec{E}_c$ toda la carga contenida en la corteza esférica.

$$\int_{\text{esf. } r\geq R} \vec{E}_c \cdot d\vec{S} = |\vec{E}_c|\,(4\pi r^2) = \frac{Q_{TOT}}{\epsilon_0} = \frac{1}{\epsilon_0}\int_{R/2}^{R} \rho_0\, dV = \frac{1}{\epsilon_0}\int_{R/2}^{R} \rho_0\,(4\pi r^2)\, dr$$

$$|\vec{E}_c|\,(4\pi r^2) = \frac{4\pi\rho_0}{\epsilon_0}\int_{R/2}^{R} r^2 dr = \frac{4\pi\rho_0}{3\epsilon_0}\left[R^3 - \frac{R^3}{8}\right] \qquad r \geq R$$

$$|\vec{E}_c|\,(4\pi r^2) = \frac{4\pi\rho_0}{3\epsilon_0}\left[R^3 - \frac{R^3}{8}\right] = \frac{4\pi\rho_0 R^3}{3\epsilon_0}\left[1 - \frac{1}{8}\right] = \frac{Q_{TOT}}{\epsilon_0}$$

$$|\vec{E}_c|\,(\cancel{4\pi}r^2) = \underbrace{\frac{\cancel{4\pi}\rho_0 R^3}{3\epsilon_0} \times \frac{7}{8}}_{\text{“}Q_{TOT}\text{”}} \rightarrow \vec{E}_c = \frac{7}{24}\frac{\rho_0 R^3}{\epsilon_0 r^2}\vec{e}_r$$

equivalentemente si escribimos

$$Q_{TOT} = \frac{4\pi\rho_0 R^3}{3} \times \frac{7}{8} = \frac{7\pi}{6}\rho_0 R^3 \quad \text{tenemos simplemente}$$

$$\vec{E}_c = \frac{Q_{TOT}}{4\pi\epsilon_0 r^2}\vec{e}_r$$ y para $r \geq R$ recuperamos el campo eléctrico creado por una carga puntual Q_{TOT} y el potencial eléctrico para $r \geq R$ será simplemente

$$\phi_c = \frac{Q_{TOT}}{4\pi\epsilon_0 r} \xrightarrow[r\to\infty]{} 0 \quad y \quad \phi_c(R) = \frac{Q_{TOT}}{4\pi\epsilon_0 R} = \frac{7\pi\rho_0 R^3/6}{4\pi\epsilon_0 R}$$

luego $\phi_c(R) = \dfrac{7\rho_0 R^2}{24\epsilon_0}$

Para calcular U_e tenemos que conocer el potencial en la región (b) que es donde $\rho(\vec{r}) \neq 0$, por tanto

$$\underbrace{\phi_c(R)}_{=\,\phi_b(R)} - \phi_b(r) = -\int_r^R \vec{E}_b \cdot d\vec{r} = -\int_r^R \frac{\rho_0}{3\epsilon_0}\left[r - \frac{R^3}{8r^2}\right] dr$$

por continuidad

$$\phi_c(R) - \phi_b(r) = -\frac{\rho_0}{3\epsilon_0}\int_r^R \left[r - \frac{R^3}{8r^2}\right] dr$$

$$\phi_c(R) - \phi(r) = -\frac{\rho_0}{3\epsilon_0}\left[\frac{r^2}{2} - \frac{R^3}{8}\frac{r^{-1}}{(-1)}\right]_r^R = -\frac{\rho_0}{3\epsilon_0}\left[\frac{r^2}{2} + \frac{R^3}{8r}\right]_r^R$$

$$\underbrace{\phi_c(R)}_{=\,\frac{7\rho_0 R^2}{24\epsilon_0}} - \phi_b(r) = -\frac{\rho_0}{3\epsilon_0}\left[\frac{R^2}{2} - \frac{r^2}{2} + \frac{R^3}{8R} - \frac{R^3}{8r}\right]$$

$$\phi_b(r) = \underbrace{\frac{\rho_0}{3\epsilon_0}\left(\frac{7R^2}{8}\right)}_{\phi_c(R)} + \frac{\rho_0}{3\epsilon_0}\left[\frac{R^2}{2} + \frac{R^2}{8} - \frac{r^2}{2} - \frac{R^3}{8r}\right]$$

$$\phi_c(r) = \frac{\rho_0}{3\epsilon_0}\left[\frac{7R^2}{8} + \frac{R^2}{2} + \frac{R^2}{8} - \frac{r^2}{2} - \frac{R^3}{8r}\right]$$

$$\phi_b(r) = \frac{\rho_0}{3\epsilon_0}\left[R^2\underbrace{\left(\frac{7}{8} + \frac{1}{8} + \frac{1}{2}\right)}_{1 + \frac{1}{2} = \frac{3}{2}} - \frac{r^2}{2} - \frac{R^3}{8r}\right]$$

$$\phi_b(r) = \frac{\rho_0}{3\epsilon_0}\left[\frac{3R^2}{2} - \frac{r^2}{2} - \frac{R^3}{8r}\right] \text{ para } \frac{R}{2} < r \leq R \quad [1]$$

y ahora podemos calcular la integral

$$U_e = \frac{1}{2}\int_{\frac{R}{2}<r\leq R} \rho\,\phi_b\,dV = \frac{1}{2}\int_{R/2}^{R} \rho_0 \times \left[\frac{\rho_0}{3\epsilon_0}\left(\frac{3R^2}{2} - \frac{r^2}{2} - \frac{R^3}{8r}\right) \times (4\pi r^2)\,dr\right.$$

$$U_e = \frac{4\pi\rho_0^2}{6\epsilon_0}\int_{R/2}^{R}\left[\frac{3R^2}{2} - \frac{r^2}{2} - \frac{R^3}{8r}\right] r^2\,dr$$

$$U_e = \frac{4\pi\rho_0^2}{6\epsilon_0}\int_{R/2}^{R}\left[\frac{3R^2 r^2}{2} - \frac{r^4}{2} - \frac{R^3 r}{8}\right] dr$$

$$U_e = \frac{4\pi\rho_0^2}{6\epsilon_0}\left[\frac{R^2 r^3}{2} - \frac{r^5}{10} - \frac{R^3 r^2}{16}\right]_{R/2}^{R}$$

$$U_e = \frac{4\pi\rho_0^2}{6\epsilon_0}\left[\left(\frac{R^5}{2} - \frac{R^5}{10} - \frac{R^5}{16}\right) - \left(\frac{R^5}{2\times 2^3} - \frac{R^5}{10\times 2^5} - \frac{R^5}{16\times 2^2}\right)\right]$$

$$U_e = \frac{4\pi\rho_0^2 R^5}{6\epsilon_0}\left[\underbrace{\left(\frac{1}{2} - \frac{1}{10} - \frac{1}{16}\right)}_{= 27/80} - \underbrace{\left(\frac{1}{2^4} - \frac{1}{10\times 2^5} - \frac{1}{\underbrace{2^4\times 2^2}_{16\times 2^2}}\right)}_{= \frac{7}{2\times 80}}\right]$$

$$U_e = \frac{4\pi\rho_0^2 R^5}{6\epsilon_0}\underbrace{\left[\frac{27}{80} - \frac{7}{2\times 80}\right]}_{= \frac{47}{2\times 80}} = \frac{4\pi\rho_0 R^5}{6\epsilon_0}\left[\frac{47}{2\times 80}\right]$$

$$U_e = \frac{\cancel{4}\pi \rho_o^2 R^5}{6\,\epsilon_0}\,\frac{47}{\cancel{4}\times 40} = \frac{\pi \rho_o^2 R^5}{\epsilon_0}\,\frac{47}{240}$$

$$U_e = \frac{47}{240}\,\frac{\pi \rho_o^2 R^5}{\epsilon_0} \quad [2]$$

Ahora podemos calcular la energía electrostática del conjunto que será.

$$U_{TOT} = Q_1 \phi(\bar{r}_1) + U_e$$

energía electrostática de la distribución calculada en [2]

Potencial eléctrico donde se encuentra la carga Q_1 excluida esta.

El potencial $\phi(r_1) = \phi_a(0)$ y como el campo eléctrico $\vec{E}_a = 0$ es una constante y por continuidad tiene que ser igual a $\phi_b(R/2) = \phi_a(0)$ que es el que da la ecuación [1] anterior,

$$\phi_b(r) = \frac{\rho_o}{3\,\epsilon_0}\left[\frac{3R^2}{2} - \frac{r^2}{2} - \frac{R^3}{8r}\right]$$

en $r = R/2$ que es entonces

$$\phi_a(0) = \phi_b(R/2) = \frac{\rho_o}{3\,\epsilon_0}\left[\frac{3R^2}{2} - \frac{R^2}{2^2\times 2} - \frac{2R^3}{8R}\right]$$

$$\phi_a(0) = \phi_b(R/2) = \frac{\rho_o R^2}{3\,\epsilon_0}\underbrace{\left[\frac{3}{2} - \frac{1}{8} - \frac{1}{4}\right]}_{=9/8}$$

$$\phi_a(0) = \frac{\rho_o R^2}{3\,\epsilon_0}\times\frac{9}{8} = \frac{3\rho_o R^2}{8\,\epsilon_0}$$

Queda entonces

$$W_{tot} = \varphi_1 \times \left(\frac{3 \rho_0 R^2}{8 \varepsilon_0} \right) + \frac{47}{240} \frac{\pi \rho_0^2 R^5}{\varepsilon_0}$$