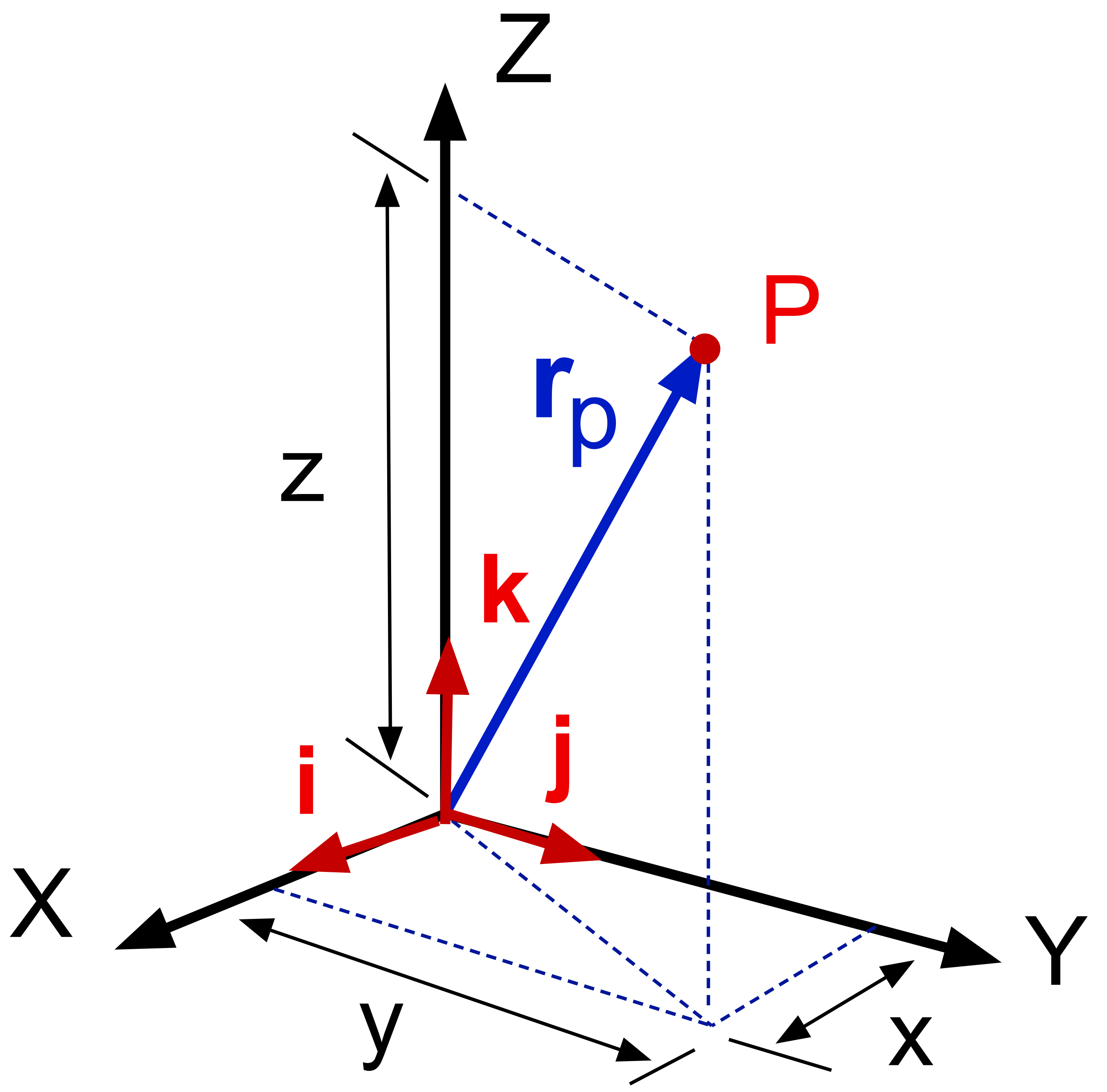
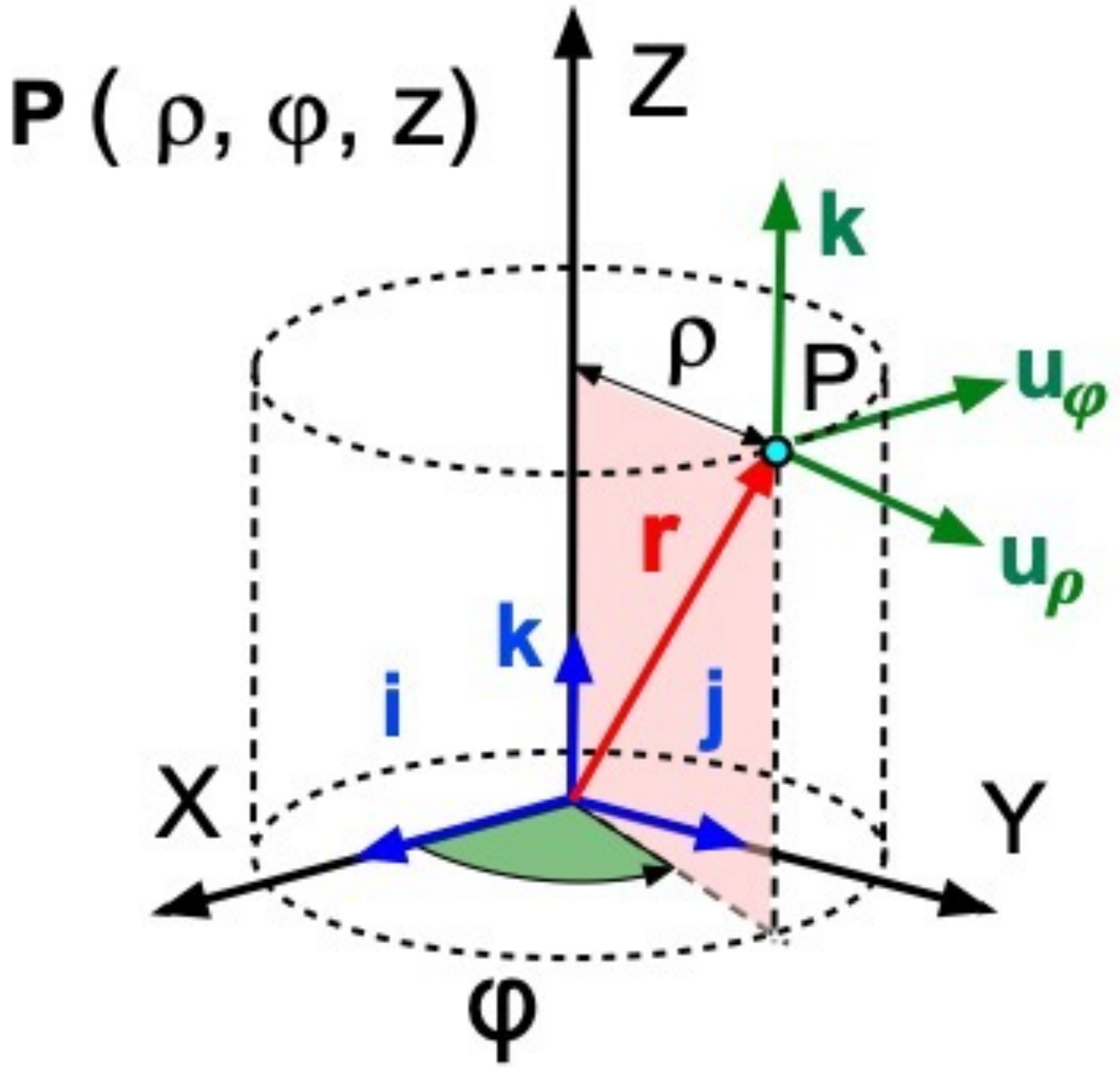
Sistemas coordenados



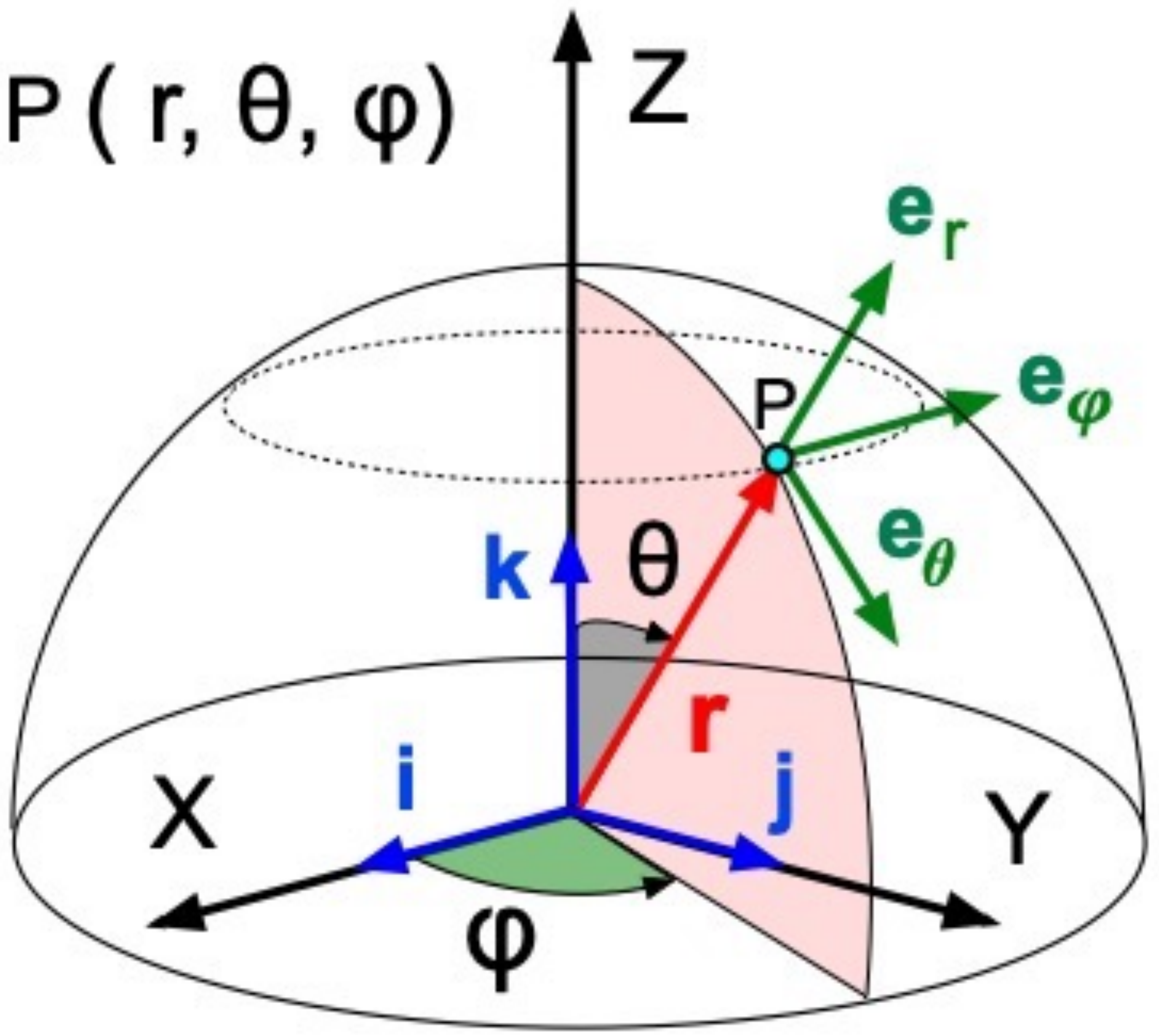
Rectangulares
o cartesianas

$$\boldsymbol{r}_p = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$$



Cilíndricas

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\,\mathrm{sen}\,\varphi \\ z = z \\ \rho = \sqrt{x^2 + y^2} \end{cases}$$



Esféricas

$$\begin{cases} x = r\,\mathrm{sen}\,\theta\cos\varphi \\ y = r\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\varphi \\ z = r\cos\theta \\ r = \sqrt{x^2 + y^2 + z^2} \end{cases}$$

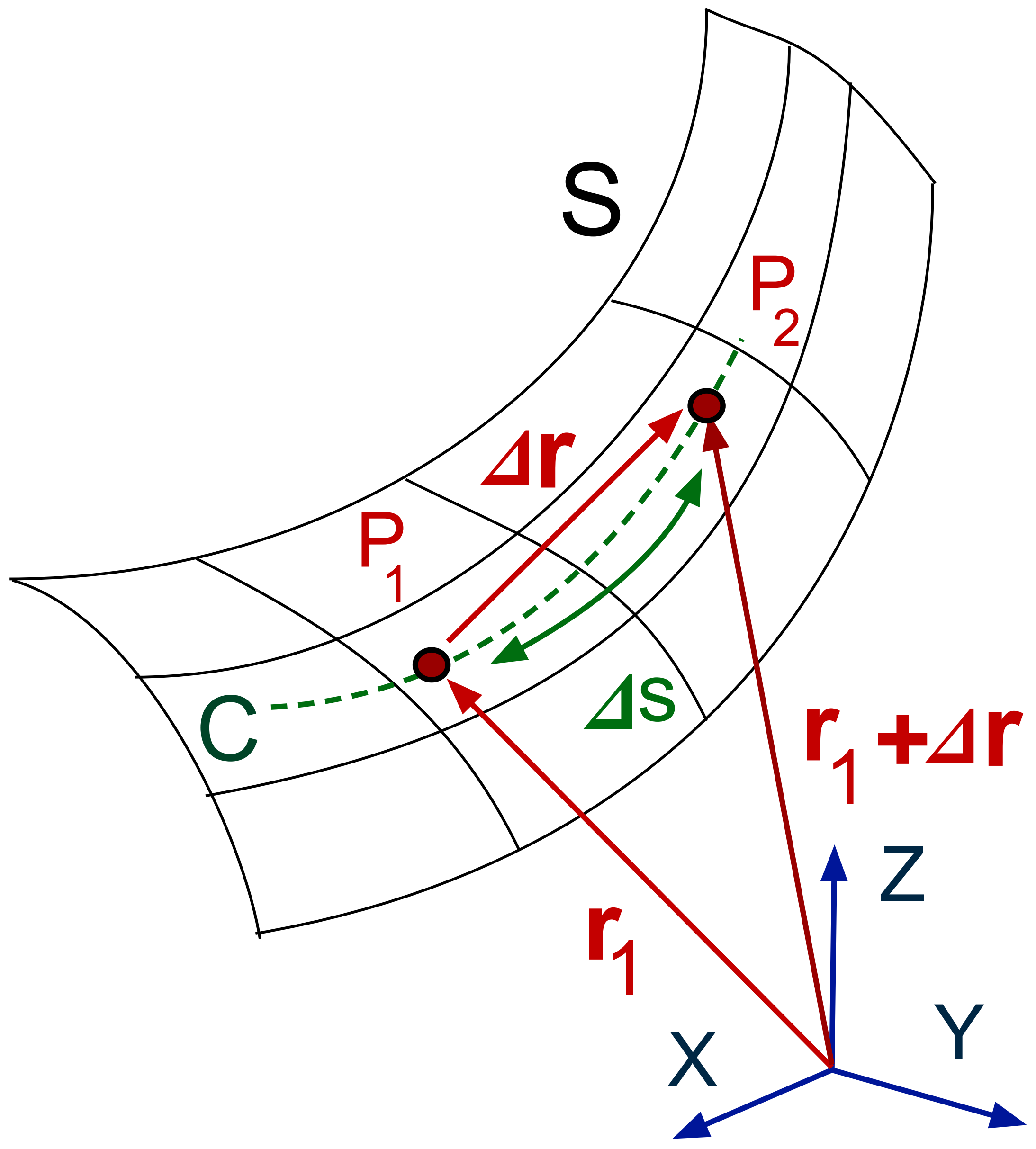
- Derivada de un vector respecto de un parámetro

$$\frac{d\boldsymbol{P}}{dt} = \lim_{\delta t \to 0} \frac{\boldsymbol{P}(t+\delta t) - \boldsymbol{P}(t)}{\delta t} = \frac{dP_x}{dt}\boldsymbol{i} + \frac{dP_y}{dt}\boldsymbol{j} + \frac{dP_z}{dt}\boldsymbol{k}$$

- Reglas de derivación: λ es un escalar $\boldsymbol{Q}(t)$ y $\boldsymbol{P}(t)$ vectores

$$\left\{\begin{aligned}
&\frac{d}{dt}(\lambda\, \boldsymbol{P}(t)) = \lambda\, \frac{d\boldsymbol{P}}{dt} \\
&\frac{d}{dt}(\, \boldsymbol{P}(t) + \boldsymbol{Q}(t)) = \frac{d\boldsymbol{P}}{dt} + \frac{d\boldsymbol{Q}}{dt} \\
&\frac{d}{dt}(\, \boldsymbol{P}(t) \cdot \boldsymbol{Q}(t)) = \frac{d\boldsymbol{P}}{dt} \cdot \boldsymbol{Q} + \boldsymbol{P} \cdot \frac{d\boldsymbol{Q}}{dt} \\
&\frac{d}{dt}(\, \boldsymbol{P}(t) \wedge \boldsymbol{Q}(t)) = \frac{d\boldsymbol{P}}{dt} \wedge \boldsymbol{Q} + \boldsymbol{P} \wedge \frac{d\boldsymbol{Q}}{dt}
\end{aligned}\right.$$

- A lo largo de una curva $\boldsymbol{r}(s)$ donde s es una longitud



$$\lim_{\Delta s \to 0} \frac{\Delta \boldsymbol{r}}{\Delta s} = \frac{d\boldsymbol{r}}{ds} = \boldsymbol{t}$$

Vector tangente a la curva

$$\boldsymbol{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t} = \frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}}{ds} \times \frac{ds}{dt} = v(t)\ \boldsymbol{t}$$

- Diferencial de una función escalar: $d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$

$$d\phi = \left[\frac{\partial \phi}{\partial x}\boldsymbol{i} + \frac{\partial \phi}{\partial y}\mathrm{j} + \frac{\partial \phi}{\partial z}\boldsymbol{k}\right] \cdot (dx\ \boldsymbol{i} + dy\ \boldsymbol{j} + dz\ \boldsymbol{k}) = \nabla\phi \circ d\boldsymbol{r}$$

A lo largo de la curva $\boldsymbol{r}(s)$ tendremos $\frac{d\phi}{ds} = \nabla\phi \cdot \frac{d\boldsymbol{r}}{ds} = \nabla\phi \cdot \boldsymbol{t}$

- Derivada direccional: $\boldsymbol{D}_{\boldsymbol{u}}\,\phi = \lim_{\Delta s \to 0} \frac{\phi(\boldsymbol{r_o} + \Delta s\ \mathbf{u}) - \phi(\boldsymbol{r_o})}{\Delta s} = \nabla\phi \cdot \boldsymbol{u}$

Operadores diferenciales:

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\boldsymbol{i}\frac{\partial}{\partial x} + \mathrm{j}\,\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right)$$

Operador nabla

- Gradiente de una función escalar: $\nabla\phi = \left(\frac{\partial\phi}{\partial x}\,\boldsymbol{i} + \frac{\partial\phi}{\partial y}\,\boldsymbol{j} + \frac{\partial\phi}{\partial z}\,\boldsymbol{k}\right)$

- Divergencia de una función vectorial:

$$\nabla \circ P = \left(\boldsymbol{i}\frac{\partial}{\partial x} + \mathrm{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right) \cdot \left(P_x\,\boldsymbol{i} + P_y\,\mathrm{j} + P_z\,\boldsymbol{k}\right) = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}$$

- Rotacional de una función vectorial:

$$\nabla \wedge \boldsymbol{P} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P_x & P_y & P_z \end{vmatrix} = \left(\frac{\partial P_z}{\partial y} - \frac{\partial P_y}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial P_z}{\partial x} - \frac{\partial P_x}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial P_x}{\partial y} - \frac{\partial P_y}{\partial x}\right)\boldsymbol{k}$$

Identidades vectoriales: las más sencillas se pueden comprobar directamente

Son $\boldsymbol{P}(r)$ y $\boldsymbol{Q}(r)$ campos vectoriales; $\phi(r)$ y $\varphi(r)$ campos escalares.

$$\nabla \cdot \boldsymbol{r} = 3$$

$$\nabla \wedge \boldsymbol{r} = 0$$

$$\nabla \wedge (\nabla \phi) = 0$$

$$\nabla \cdot (\nabla \wedge \boldsymbol{P}) = 0$$

$$\nabla(\phi\ \varphi) = (\nabla\phi)\ \varphi + \phi\ (\nabla\varphi)$$

$$\nabla \wedge (\phi\ \boldsymbol{P}) = (\nabla\phi) \wedge \boldsymbol{P} + \phi\ (\nabla \wedge \boldsymbol{P})$$

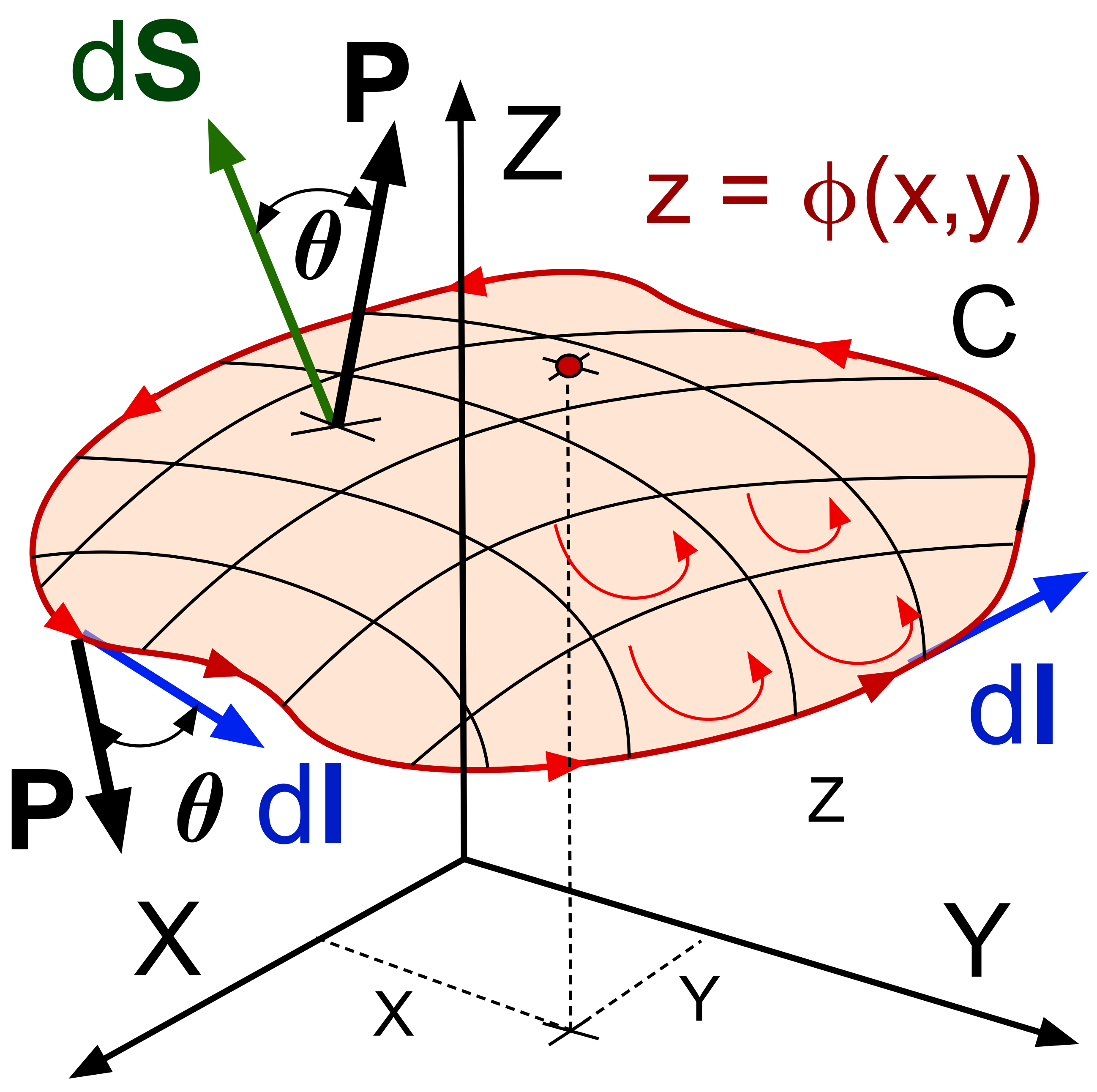
$$\nabla \wedge (\nabla \wedge \boldsymbol{P}) = \nabla(\nabla \cdot \boldsymbol{P}) - \nabla^2 \boldsymbol{P}$$

$$\nabla \cdot (\boldsymbol{P} \wedge \boldsymbol{Q}) = \boldsymbol{Q} \cdot (\nabla \wedge \boldsymbol{P}) - \boldsymbol{P} \cdot (\nabla \wedge \boldsymbol{Q})$$

Expresiones de los operadores diferenciales en los sistemas coordenados

	Cartesianas	Cilíndricas	Esféricas
Transformación		$x = \rho \cos\varphi$ $y = \rho \operatorname{sen}\varphi$ $z = z$	$x = r \operatorname{sen}\theta \cos\varphi$ $y = r \operatorname{sen}\theta \operatorname{sen}\varphi$ $z = r \cos\theta$
Elemento de longitud	$dl = \sqrt{dx^2 + dy^2 + dz^2}$	$dl = \sqrt{d\rho^2 + \rho^2\, d\varphi^2 + dz^2}$	$dl = \sqrt{dr^2 + r^2 \sin^2\theta\, d\varphi^2 + r^2\, d\theta^2}$
Elemento de volumen	$dV = dx\, dy\, dz$	$dV = \rho\, d\rho\, d\varphi\, dz$	$dV = r^2 \operatorname{sen}\theta\, dr\, d\theta\, d\varphi$
Elemento de superficie	$dS_x = dy\, dz$ $dS_y = dx\, dz$ $dS_z = dx\, dy$	$dS_\rho = \rho\, d\varphi\, dz$ $dS_\varphi = d\rho\, dz$ $dS_z = \rho\, d\rho\, d\varphi$	$dS_r = r^2 \operatorname{sen}\theta\, d\theta\, d\varphi$ $dS_\varphi = r\, dr\, d\theta$ $dS_\theta = r \operatorname{sen}\theta\, dr\, d\varphi$
Gradiente	$\nabla\phi = \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k}$	$\nabla\phi = \frac{\partial\phi}{\partial\rho}\boldsymbol{u}_\rho + \frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}\boldsymbol{u}_\varphi + \frac{\partial\phi}{\partial z}\boldsymbol{k}$	$\nabla\phi = \frac{\partial\phi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\boldsymbol{e}_\theta + \frac{1}{r \operatorname{sen}\theta}\frac{\partial\phi}{\partial\varphi}\boldsymbol{e}_\varphi$
Divergencia	$\nabla\cdot\boldsymbol{P} = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}$	$\nabla\cdot\boldsymbol{P} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho P_\rho) + \frac{1}{\rho}\frac{\partial P_\varphi}{\partial\varphi} + \frac{\partial P_z}{\partial z}$	$\nabla\cdot\boldsymbol{P} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P_r\right) + \frac{1}{r \operatorname{sen}\theta}\frac{\partial P_\varphi}{\partial\varphi} + \frac{1}{r \operatorname{sen}\theta}\frac{\partial}{\partial\theta}(\operatorname{sen}\theta\, P_\theta)$
Rotacional	$\nabla\wedge\boldsymbol{P} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P_x & P_y & P_z \end{vmatrix}$	$\nabla\wedge\boldsymbol{P} = \frac{1}{\rho}\begin{vmatrix} \boldsymbol{u}_\rho & \rho\,\boldsymbol{u}_\varphi & \boldsymbol{k} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ P_\rho & \rho P_\varphi & P_z \end{vmatrix}$	$\nabla\wedge\boldsymbol{P} = \frac{1}{r^2 \operatorname{sen}\theta}\begin{vmatrix} \boldsymbol{e}_r & r\,\boldsymbol{e}_\theta & r \operatorname{sen}\theta\,\boldsymbol{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ P_r & r P_\theta & r \operatorname{sen}\theta\, P_\varphi \end{vmatrix}$
Laplaciano	$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$	$\nabla^2\phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}$	$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2 \operatorname{sen}^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{1}{r^2 \operatorname{sen}\theta}\frac{\partial}{\partial\theta}\left(\operatorname{sen}\theta\frac{\partial\phi}{\partial\theta}\right)$

Integrales vectoriales

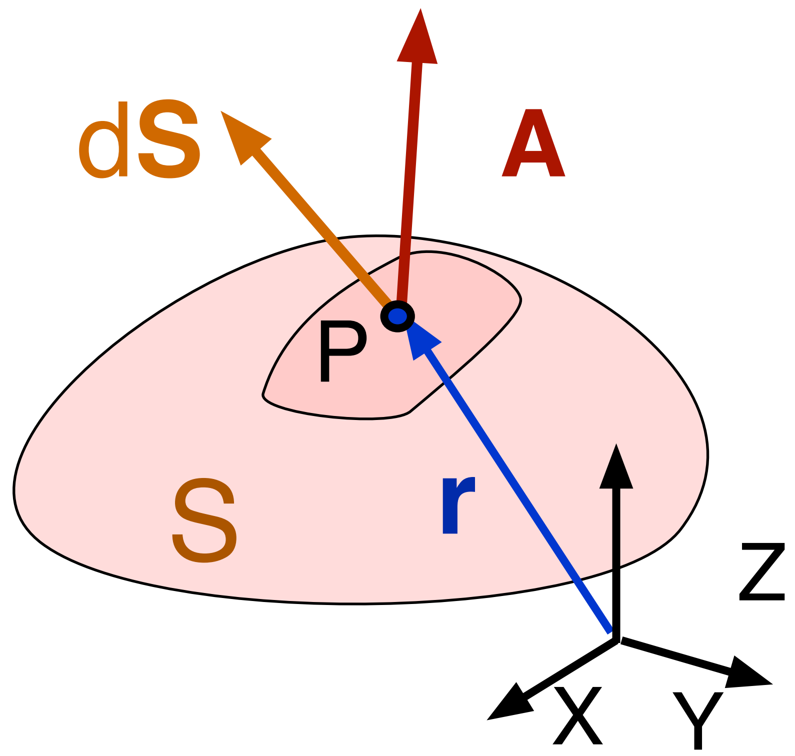


$$I = \int_C \boldsymbol{P} \cdot d\boldsymbol{l} \qquad I = \oint_C \boldsymbol{P} \cdot d\boldsymbol{l}$$

Integral de línea o circulación de un campo vectorial

$$I = \iint_C \boldsymbol{P} \cdot d\boldsymbol{s} \qquad I = \oiint_S \boldsymbol{P} \cdot d\boldsymbol{s}$$

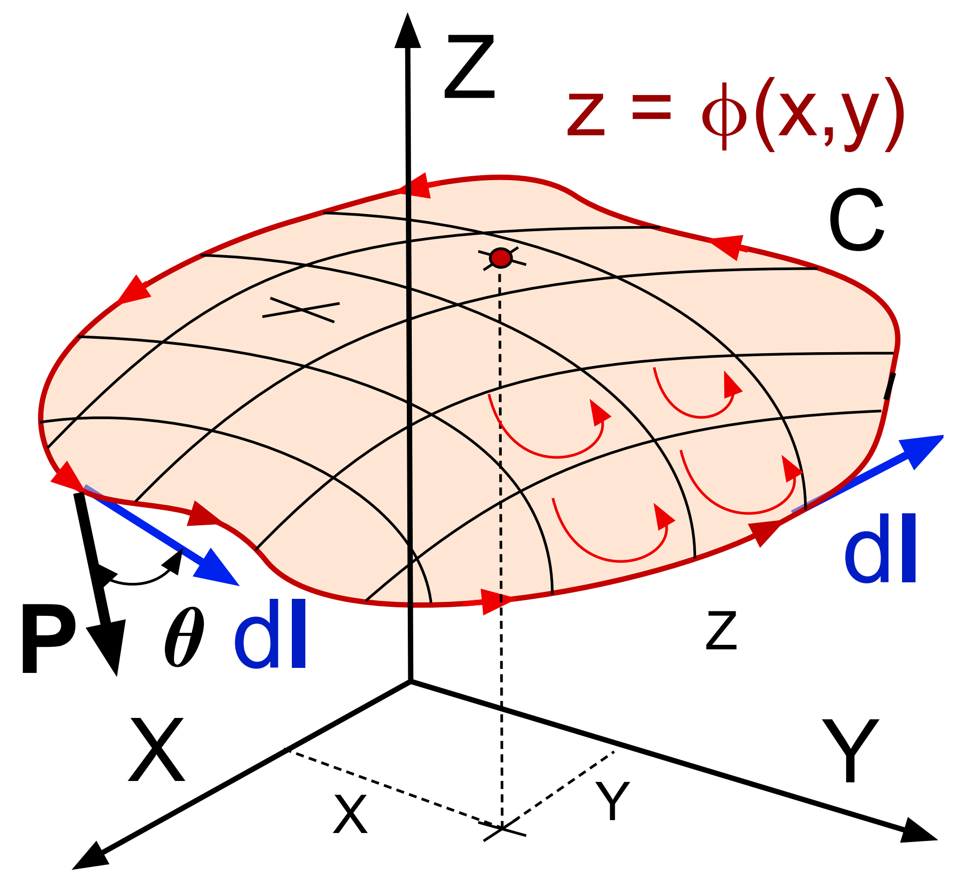
Flujo de un campo vectorial sobre una superficie



Teoremas integrales para un campo $\boldsymbol{A}(\boldsymbol{r})$ vectorial

$$\oiint_S \boldsymbol{A} \cdot d\boldsymbol{s} = \iiint_{V(S)} (\nabla \cdot \boldsymbol{A})\, dV$$ Gauss

$$\oint_C \boldsymbol{A} \cdot d\boldsymbol{l} = \oiint_{S(C)} (\nabla \wedge \boldsymbol{A}) \cdot d\boldsymbol{s}$$ Stokes



Densidad de carga: Carga eléctrica contenida por unidad de volumen.

Distribución de cargas discreta

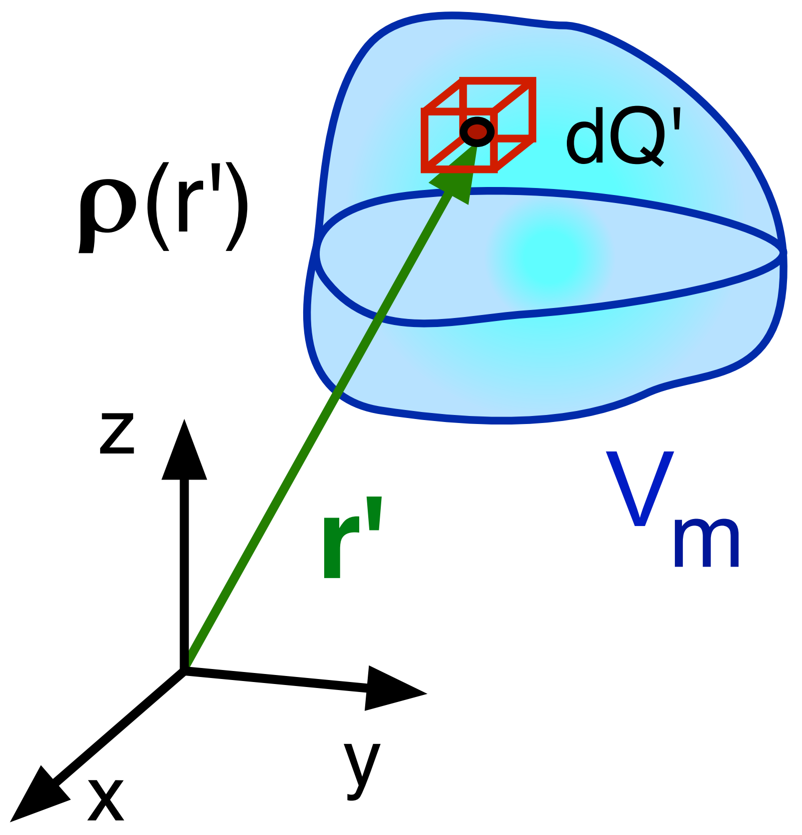
$$\rho_c(\boldsymbol{r}') = \lim_{\Delta V' \to 0} \frac{1}{\Delta V'} \sum_{j=1}^{N} q_j$$

Distribución de cargas continua

$$\rho_c(\boldsymbol{r}') = \lim_{\Delta V' \to 0} \frac{\Delta Q'}{\Delta V'} = \frac{dQ'}{dV'}$$

$$Q = \int_{V_m} \rho_c(\boldsymbol{r}')\, dV'$$

$$Q = \int_{V_m} \rho_c(\boldsymbol{r}')\, dx'\, dy' dz'$$



Densidad de corriente: Carga eléctrica que pasa en la unidad de tiempo por unidad de superficie

Distribución de cargas *discreta*

$$\boldsymbol{J}_c(\boldsymbol{r}') = \lim_{\Delta V' \to 0} \frac{1}{\Delta V'} \sum_{j=1}^{N} q_j\, \boldsymbol{u}_j$$

Distribución de cargas *continua*

$$\boldsymbol{J}_c = \rho_c(\boldsymbol{r}')\, \boldsymbol{u}(\boldsymbol{r}')$$

Superficie S abierta

$$\frac{dQ}{dt} = \int_S \boldsymbol{J}_c \cdot d\boldsymbol{s} = \int_S (\rho_c \boldsymbol{u}) \cdot d\boldsymbol{s}$$

Superficie S cerrada

$$\frac{dQ}{dt} = -\oiint_S \boldsymbol{J}_c \cdot d\boldsymbol{s}$$