The ideal Maxwellian plasma

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Plasmas are, ...

The plasma state of matter may be defined as a mixture of positively charged ions, electrons and neutral atoms which constitutes a *macroscopic electrically neutral medium* which *responds to the electric and magnetic fields in a collective mode*

Properties

- Charged particles interact through *long distance electromagnetic forces* in addition to short range molecular collisions.
- The density of negative n_e and positive n_i charged particles are equal, so that on the average *the medium is electrically neutral* (quasineutrality).
- The response to external perturbations is collective, large number of charges are involved.
- We may have *multicomponent plasmas*, ions with negative charge, dusty plasmas (complex plasmas), ...etc.

Practical considerations ...

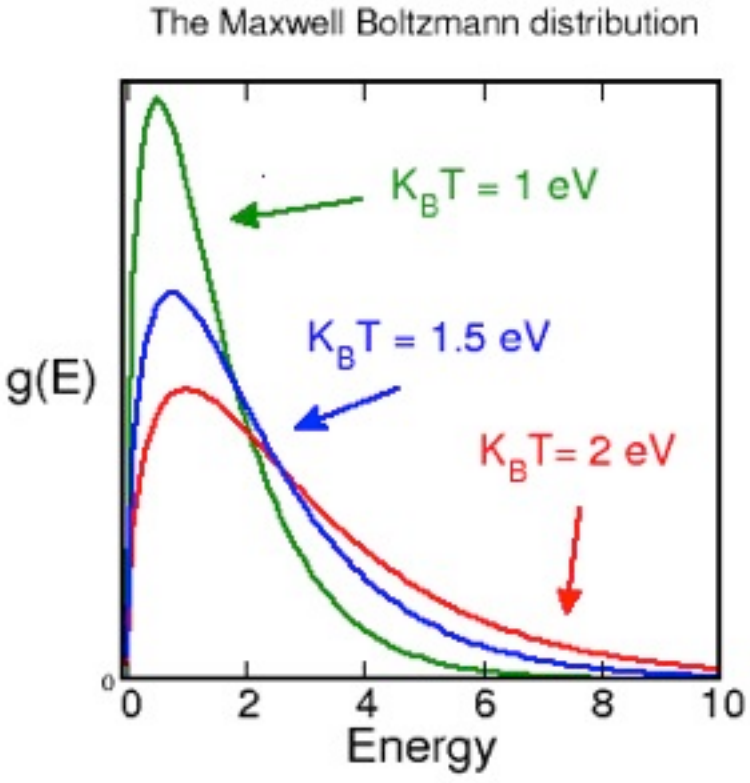
- For simplicity, we will limit to one component classical plasmas with single charged ions.
- The particles in our plasmas α= a, i, e will be electrons e , ions i and eventually neutral atoms a with number densities n_α
- The particle temperatures equivalent to the average kinetic energy of particles are usually expressed in eV and 1 eV = 11600 K,

$$\frac{e}{k_B T} = \frac{1.6 \times 10^{-19}}{1.38 \times 10^{-23} \times 1,0} = 11.594,2 \simeq 11.600\,\mathrm{K}$$

- The ion charge is $Q_i = eZ$ but in most cases we will consider only single charged ions (Z=1).
- We will use MKSC unit system.

The equilibrium state of a neutral gas, ...

The particle energy distribution function of equilibrium system is Maxwellian with temperature $k_B T$



$$\begin{cases} f_a(\mathbf{v}) = \left(\dfrac{m_a}{2\pi\, k_B T}\right)^{3/2} \exp\left(-\dfrac{m_a \mathbf{v}^2}{2 k_B T}\right) \\[2ex] g_a(E) = \dfrac{2}{\sqrt{\pi}} \dfrac{\sqrt{E}}{(k_B T)^{3/2}} \exp\left(-E/k_B T\right) \end{cases}$$

The normalized probability distribution

$$\int_0^{\infty} g_a(E)\, dE = \int_{-\infty}^{+\infty} f_a(\mathbf{v})\, d\mathbf{v} = 1$$

$$\left.\begin{aligned} dn_a &= n_{ao}\, f_a(\mathbf{v})\, d\mathbf{v} \\ dn_a &= n_{ao}\, g_a(E)\, dE \end{aligned}\right\} \quad dn_a \text{ particles with } \begin{cases} (\mathbf{v},\ \mathbf{v} + d\mathbf{v}) \\ (E,\ E + dE) \end{cases}$$

The macroscopic properties, ...

The macroscopic physical properties are calculated as averages of the particle energy distribution function. The average energy per particle,

$$\langle E \rangle = \int_0^\infty E\, g_a(E)\, dE = \frac{3}{2}\, k_B T \qquad E_i = n_{ao}\, \langle E \rangle = \frac{3}{2}\, n_{ao}\, k_B T$$

The thermal speed redefines the velocity distribution function,

$$v_{th} = \sqrt{\frac{2\, k_B T}{m_a}} \qquad f_a(\mathbf{v}) = \frac{1}{(\sqrt{\pi}\, v_{th})^3}\, \exp\left(-\mathbf{v}^2 / v_{th}^2\right)$$

The velocity averages, $\langle v_x \rangle = \langle v_y \rangle = \langle v_z \rangle = 0$

$$v_{ave} = \langle \sqrt{\mathbf{v} \cdot \mathbf{v}} \rangle = \sqrt{\frac{8 k_B T}{\pi\, m_a}} \qquad \langle |v_x| \rangle = \sqrt{\frac{2\, k_B T}{\pi\, m_a}}$$

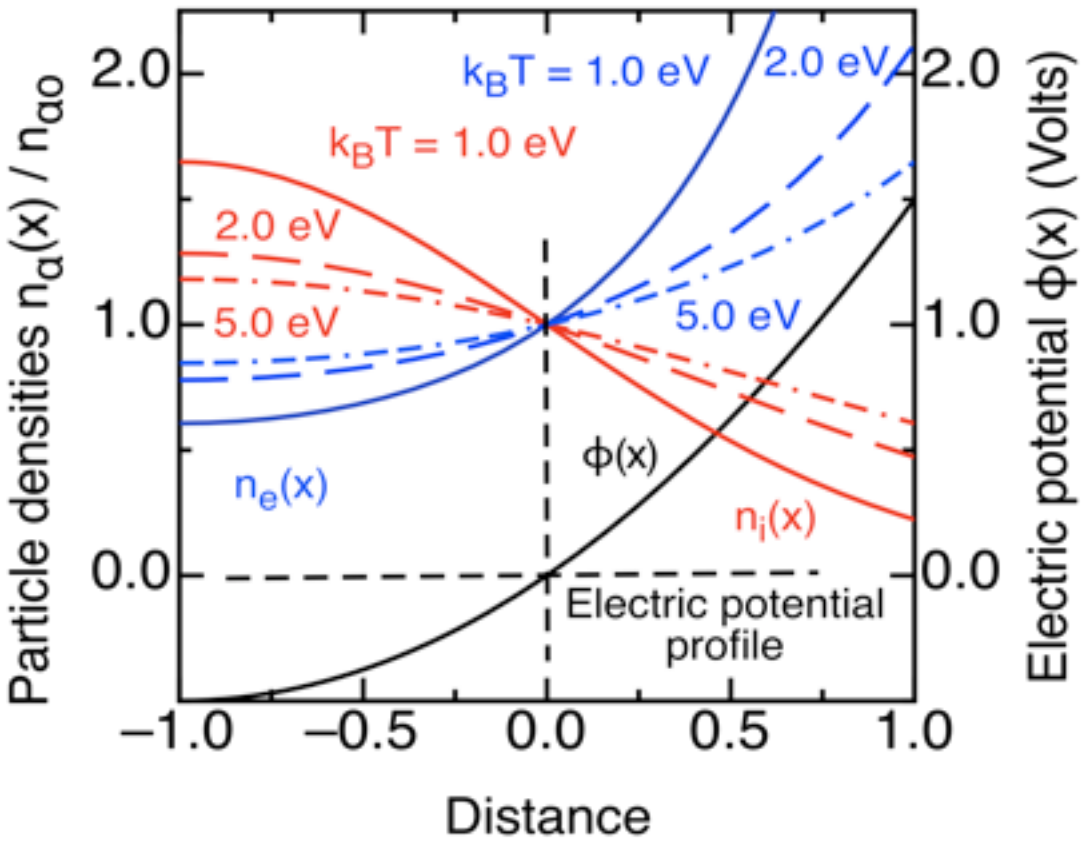
define the particle flux, $\Gamma_a = \frac{1}{2}\, n_{ao}\, v_{ave} \qquad J_\alpha = q_\alpha\, \Gamma_\alpha = q_\alpha\, \frac{1}{2}\, n_{ao}\, v_{ave}$

The equilibrium of a ideal Maxwellian plasma, ...

- The energy distribution function of charged particles ions and electrons is Maxwellian.
- All particle groups have the same kinetic temperature $K_B T_e = K_B T_i = K_B T$ and $K_B T$ is the temperature corresponding to the thermodynamic equilibrium.
- Plasma quasineutrality, $n_e \approx n_i$ implies that no electric fields exists in the equilibrium plasma bulk; uniform plasma potential.
- No transport; no particle currents.

The Maxwellian plasma in a external electric field ...

Background equilibrium, $n_{eo} \simeq n_{io}$



$$E_\alpha = \frac{m_\alpha \mathbf{v}^2}{2} + q_\alpha \phi(x)$$

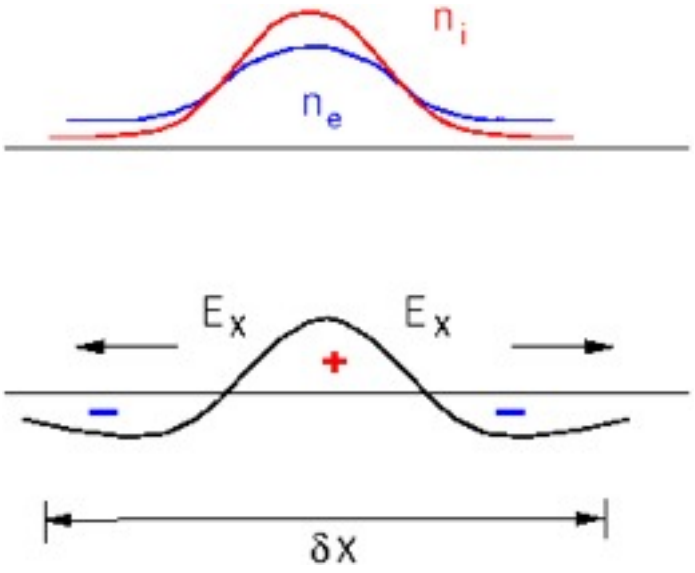
$$n_\alpha = n_{\alpha o} \int_0^\infty g_\alpha(E)\, dE$$

$$\begin{cases} n_e(x) = n_{eo} \exp\left(\dfrac{e\phi}{k_B T}\right) \\ n_i(x) = n_{io} \exp\left(-\dfrac{e\phi}{k_B T}\right) \end{cases}$$

In *cold plasma* $k_BT \approx 0$ the charge separation is complete while in a *finite temperature plasma* $k_BT \neq 0$ the thermal and electrostatic energies compete. This permits *to shield out* the small amplitude fluctuations of electric and magnetic fields.

The length and time scales, ...

The damping and/or attenuation of small amplitude fluctuations of the equilibrium takes place over a characteristic length and time scales.



- Space fluctuations: Debye length
- Time fluctuations: Plasma frequency
- Number of charges: Plasma parameter

$$\lambda_{De} = \sqrt{\frac{\epsilon_o\, k_B T}{e^2\, n_{eo}}} \qquad \lambda_{Di} = \sqrt{\frac{\epsilon_o\, k_B T}{e^2\, n_{io}}}$$

$$k_B T_e \gg k_B T_i \qquad \lambda_{De} \gg \lambda_{Di}$$

$$\omega_{pi} = \sqrt{\frac{e^2\, n_{io}}{\epsilon_o\, m_i}} \qquad \omega_{pe} = \sqrt{\frac{e^2\, n_{eo}}{\epsilon_o\, m_e}} \qquad \frac{\omega_{pi}}{\omega_{pe}} = \sqrt{\frac{m_e}{m_i}}$$

The Debye length and shielding, ...

The initial equilibrium: $\mathbf{E}_o = 0 \quad n_{eo} \simeq n_{io} = n_o \quad k_B T_e; k_B T_i$

where we introduce an small perturbation in the electric charge,

$$\left.\begin{aligned} \delta\rho_{ext} &= q\,\delta(\mathbf{r}) \\ \delta\rho_{sp} &= e\left[n_i(\mathbf{r}) - n_e(\mathbf{r})\right] \end{aligned}\right\} \quad \mathbf{E}(\mathbf{r}) \simeq \mathbf{E}_o + \mathbf{E}_1(\mathbf{r}) \qquad \mathbf{E}_1(\mathbf{r}) = -\nabla\varphi_1(\mathbf{r})$$

and ***E**$_1$(**r**)* represents the *perturbed electric field*.

The field ***E**$_1$(**r**)* is governed by the Poisson equation, $\nabla \cdot \mathbf{E} = \dfrac{\delta\rho_{ext} + \delta\rho_{sp}}{\epsilon_o}$

$$\nabla \cdot \mathbf{E}_1 = \frac{q}{\epsilon_o}\delta(\mathbf{r}) + \frac{e}{\epsilon_o}\left[n_i(\mathbf{r}) - n_e(\mathbf{r})\right]) \quad \text{where,} \quad n_\alpha(\mathbf{r}) = n_{\alpha o}\,\exp\left(\pm\frac{e\,\varphi_1(\mathbf{r})}{k_B T_\alpha}\right)$$

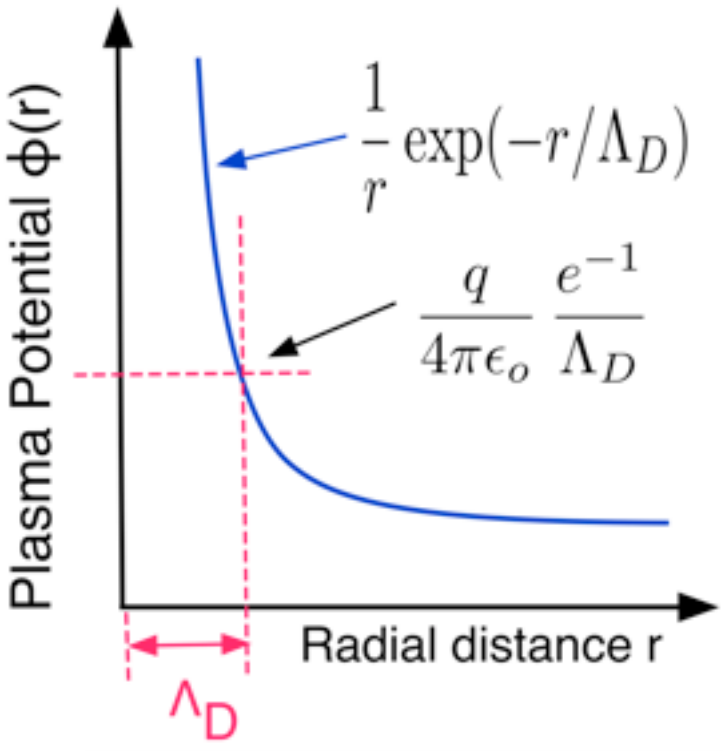
Small amplitude perturbations of the charge/electric field means that the *thermal energy* |e φ(***r***)| dominates over the *electrostatic energy* $k_B T$

$$\alpha = e, i \quad \left|\frac{e\,\varphi_1(\mathbf{r})}{k_B T_\alpha}\right| \ll 1 \quad \text{we approximate} \quad n_\alpha(\mathbf{r}) \simeq n_{\alpha o}\left(1 \pm \frac{e\,\varphi_1(\mathbf{r})}{k_B T_\alpha}\right)$$

$$-\nabla^2\varphi_1(\mathbf{r}) = \frac{1}{\epsilon_o}\left[\delta\rho_{ext} + \left(\frac{e^2\,n_o}{k_B T_e}\,\varphi_1(\mathbf{r})\right) + \left(\frac{e^2\,n_o}{k_B T_i}\,\varphi_1(\mathbf{r})\right)\right] \qquad \alpha = e, i$$

that introduces the Debye lengths, $\lambda_{D\alpha} = \sqrt{\dfrac{\epsilon_o k_B T_\alpha}{e^2\,n_o}}$ Setting,

$$\frac{1}{\Lambda^2} = \frac{1}{\lambda_{De}^2} + \frac{1}{\lambda_{Di}^2} \qquad \left(\nabla^2 - \frac{1}{\Lambda^2}\right)\varphi_1(\mathbf{r}) = -\frac{q}{\epsilon_o}\delta(\mathbf{r}) \qquad \varphi_1(\mathbf{r}) = \frac{q}{4\pi\epsilon_o}\frac{e^{-r/\Lambda}}{r}$$



The perturbations of the electric charge and/or plasma potential decay in space with an exponential rate governed by λ_D

The Debye length accounts for thermal effects, the size $\sim\lambda_D$ of the perturbed region increases with $k_B T$

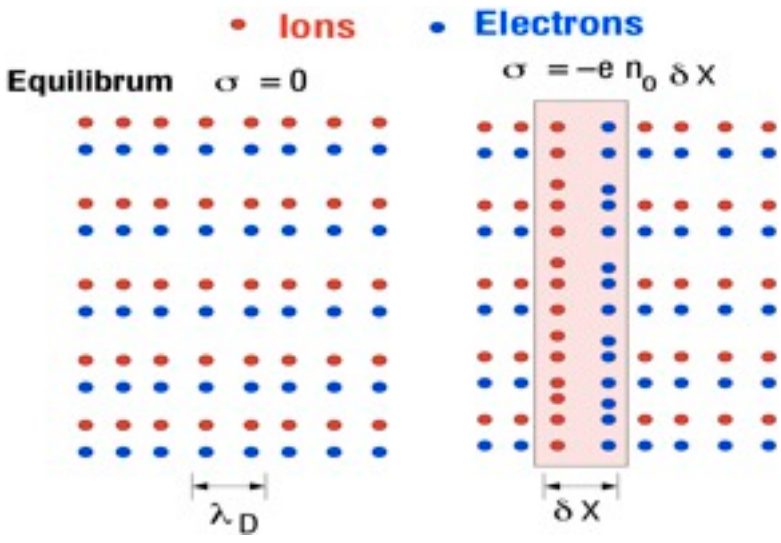
This is a linearization only valid whereas,

$$\left|\frac{e\,\varphi_1(\mathbf{r})}{k_B T_\alpha}\right| \ll 1$$

We deal with a non linear Poisson equation otherwise,

The plasma frequency, ...

Again the initial equilibrium: $\mathbf{E}_o = 0 \quad n_{eo} \simeq n_{io} = n_o \quad k_B T_e; k_B T_i$

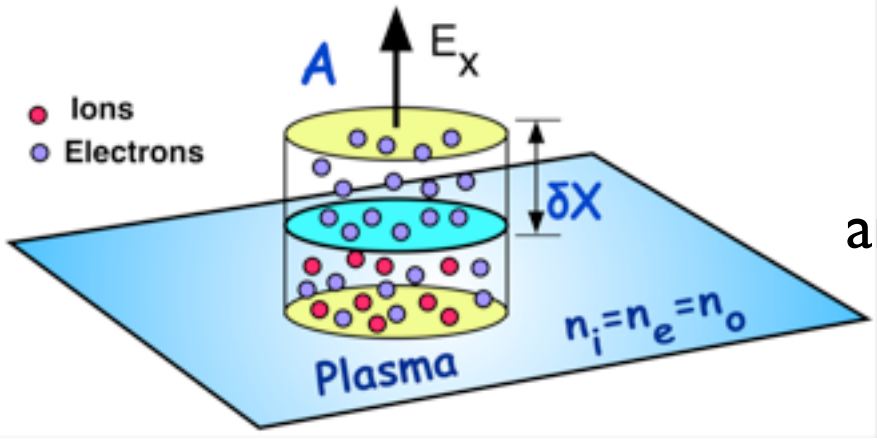


and we consider a one small perturbation of the electric charge in one dimension as in the figures, $\rho = -e\, n_o (A\, \delta X)$

$$\int_S \mathbf{E}_1 \cdot d\mathbf{s} = A\, E_{1x} = \frac{\rho}{\epsilon_o} = -\frac{e}{\epsilon_o} n_o (A\, \delta X)$$

The electric field, $E_{1x} = -\frac{e}{\epsilon_o} n_o\, \delta X$

gives us an equation of motion for ions or electrons



$$\frac{d^2}{dt^2}(\delta X) + \left[\frac{e^2 n_o}{m_\alpha \epsilon_o}\right] (\delta X) = 0$$

and defines the electron/ion plasma frequencies

$$\alpha = e, i \qquad \omega_{p\alpha} = \sqrt{\frac{e^2\, n_o}{m_\alpha\, \epsilon_o}}$$

The plasma length and time scales, ...

The Debye length λ_D (also called debye radius) determines the spatial scale where the small amplitude perturbations of the electric field are shielded out.

The plasmas have two time scales; ion motion $\tau_{pi} = 1/f_{pi}$ the (slow) and the electron response $\tau_{pe} = 1/f_{pe}$ (fast). The *ion motion is frozen* over the time scale $\tau_{pe} = 1/f_{pe}$ the electron time scale represents the faster plasma response,

$$\frac{\omega_{ie}}{\omega_{pe}} = \sqrt{\frac{m_e}{m_i}} = \sqrt{\frac{1}{1840 \times 40}} = \frac{1}{271} = 3.69 \times 10^{-3} \qquad \text{(Argon)}$$

Alternatively, the time scale the $\tau_{pe} = 1/f_{pe}$ is proportional to the time that a thermal electron travels along a Debye length,

$$\frac{\lambda_{De}}{V_{th}} = \frac{\sqrt{\epsilon_o k_B T_e / e^2 n_o}}{\sqrt{2\, k_B T_e / m_e}} = \left(\frac{\epsilon_o\, m_e}{2\, n_o\, e^2} \right)^{1/2} = \frac{1}{\sqrt{2}\, \omega_{pe}} = \frac{\tau_{pe}}{2\, \pi \sqrt{2}}$$

Approximate magnitudes in some typical plasmas ...

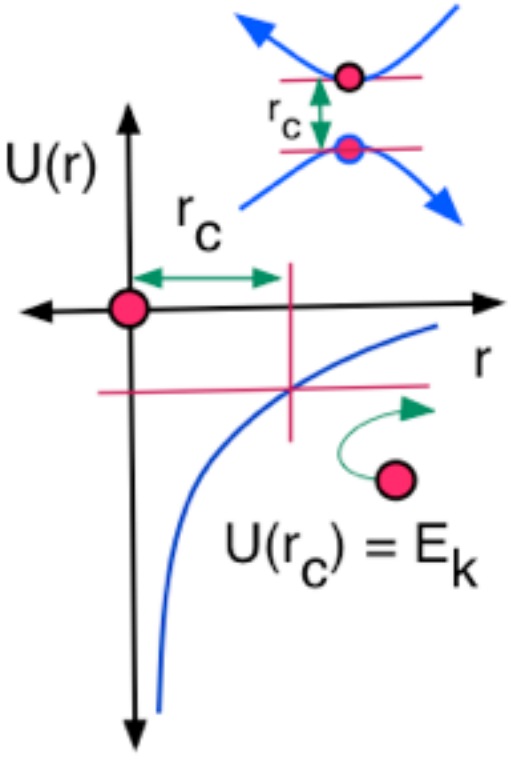
Plasma	n_e (cm^{-3})	$K_B T_e$ (eV)	λ_{De} (cm)	f_{pe} (Hz)	$n_e \lambda_{De}^3$
Interstellar gas	1	1	700	$6{,}0 \times 10^4$	4×10^8
Solar corona	10^9	100	0,2	$2{,}0 \times 10^9$	8×10^6
Solar atmosphere *Gas discharge*	10^{14}	1	$7{,}0 \times 10^{-5}$	$6{,}0 \times 10^{11}$	40
Tokamak	10^{14}	10^4	$2{,}0 \times 10^{-3}$	$2{,}0 \times 10^{12}$	6×10^6

Values from *NRL Plasma Formulary 2007*. This practical reference could be downloaded for free at; http://wwwppd.nrl.navy.mil/nrlformulary/

The plasma and coupling parameters, ...

In order to shield out the electric field perturbations a minimum number of charges are needed within a sphere of radius λ_D. This defines the plasma parameter N_D for ions and electrons,

$$N_{De} = \frac{4\pi}{3}\, n_o\, \lambda_{De}^3 \qquad k_B T_e \gg k_B T_i \qquad \lambda_{De} \gg \lambda_{Di} \qquad N_{De} \gg N_{Di}$$



The plasma parameter is closely related with *the coupling parameter* Γ_C the ratio between thermal and electrostatic energies.

$$\Gamma_C = \frac{r_c}{r_d}$$

$r_d \sim n_o^{-1/3}$ average packing of charges

r_c is the closest colliding approach as in the figure

$$E(\mathbf{r}, \mathbf{v}_\alpha) = \frac{m_\alpha\, v_\alpha^2}{2} - \frac{e^2}{4\pi\epsilon_o\, r} \qquad E(r_c, v_{th}) = 0$$

$$r_c = \frac{e^2}{4\pi\epsilon_o k_B T} \qquad \Gamma_C = \frac{e^2\, n_o^{1/3}}{4\pi\, \epsilon_o\, k_B T}$$

For the coupling parameter we also have,

$$\Gamma_c^3 = \frac{1}{(4\pi)^3} \times \frac{1}{n_o^2} \times \left(\frac{n_o\, e^2}{\epsilon_o\, k_B T}\right)^3 = \frac{1}{(4\pi)^3}\frac{1}{n_o^2\,\lambda_D^6} = \frac{1}{(4\pi \times 9)}\left(\frac{3}{4\pi\, n_o \lambda_D^3}\right)^2$$

and we obtain, $\Gamma_c = \dfrac{1}{36\,\pi\, N_D^2}$ equivalent to, $\Gamma_C = \dfrac{e^2\, n_o^{1/3}}{4\pi\,\epsilon_o\, k_B T}$

Description	*Plasma parameter*	
Limits	$\Gamma_C >> 1$ $(N_D << 1)$	$\Gamma_C >> 1$ $(N_D >> 1)$
Coupling	Strongly coupled	Weakly coupled
Debye sphere	Sparsely populated	Densely populated
Electrostatic influence	Strong	Weak
Characteristic	Cold and dense	Hot and diffuse
Examples	Laser ablation plasmas Inertial fusion experiments White dwarfs Neutron stars	Ionospheric plasmas Magnetic fusion experiments Space plasmas Electric discharge plasmas

Magnetized plasmas , ...

The force experienced by electric charges in this course,

$$\alpha = e, i \qquad \mathbf{F}_\alpha = q_\alpha \, n_\alpha \, (\mathbf{E} + \mathbf{v}_\alpha \wedge \mathbf{B})$$

this gives the cyclotron frequency, $\Omega_\alpha = q_\alpha \, B_\perp \, / \, m_\alpha$ and using the perpendicular speed to the magnetic field lines we obtain the *Larmor radius*

$r_{L,\alpha} = \dfrac{m_\alpha \, v_\perp}{|q_\alpha| \, B}$ a new length is defined using the particle thermal speed

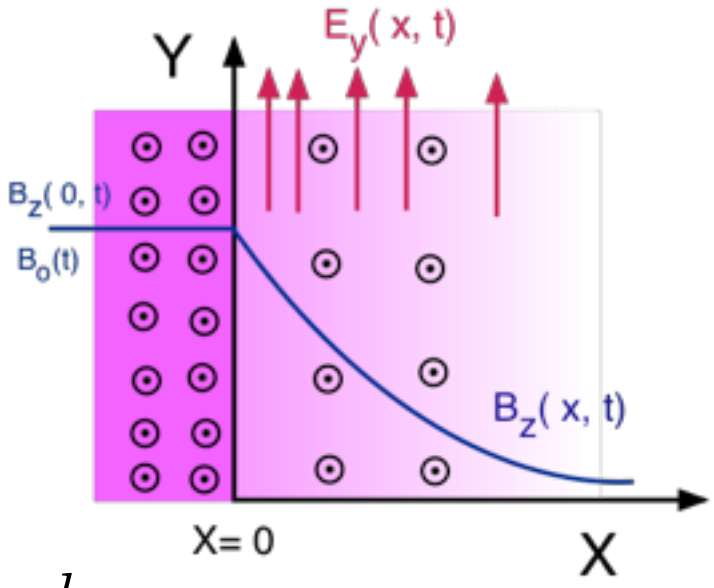
$$R_{L,\alpha} = \frac{V_{th,\alpha}}{\Omega_\alpha}$$

$$R_{L,i} = \sqrt{\frac{m_i}{m_e}} \, R_{L,e}$$

Magnetized electrons and unmagnetized ions $\quad R_{L,e} \ll R_{L,i} > L$

Magnetized electrons and ions $\quad R_{L,e} \ll R_{L,i} < L$

The plasma skin depth , ...



We consider the plasma electrons

$$\left.\begin{aligned} \mathbf{B} &= B_z(x,t)\,\mathbf{u}_z \\ B_z(0,t) &= B_o(t) \end{aligned}\right\} \qquad \begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= -\nabla \wedge \mathbf{E} \\ \mathbf{E}(x,t) &= E_y(x,t)\,\mathbf{u}_y \end{aligned}$$

We also have,

$$\nabla \wedge \mathbf{B} = \mu_o \mathbf{J_e} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \simeq -e\, n_{eo}\, \mathbf{v}_e$$

and the equation of motion for electrons

$$\left.\begin{aligned} \frac{dv_{ey}}{dt} &= -\frac{e}{m_e} E_y(x,t) \\ \frac{\partial B_z}{\partial x} &= e\,\mu_o\, n_{eo}\, \mathbf{v}_e \\ \frac{\partial B_z}{\partial t} &= -\frac{\partial E_y}{\partial x} \end{aligned}\right\}$$

$$\frac{\partial^2 E_y}{\partial x^2} = \frac{e^2}{m_e} n_{eo}\mu_o\, E_y = \frac{1}{c^2}\left(\frac{e^2\, n_{eo}}{\epsilon_o\, m_e}\right) E_y$$

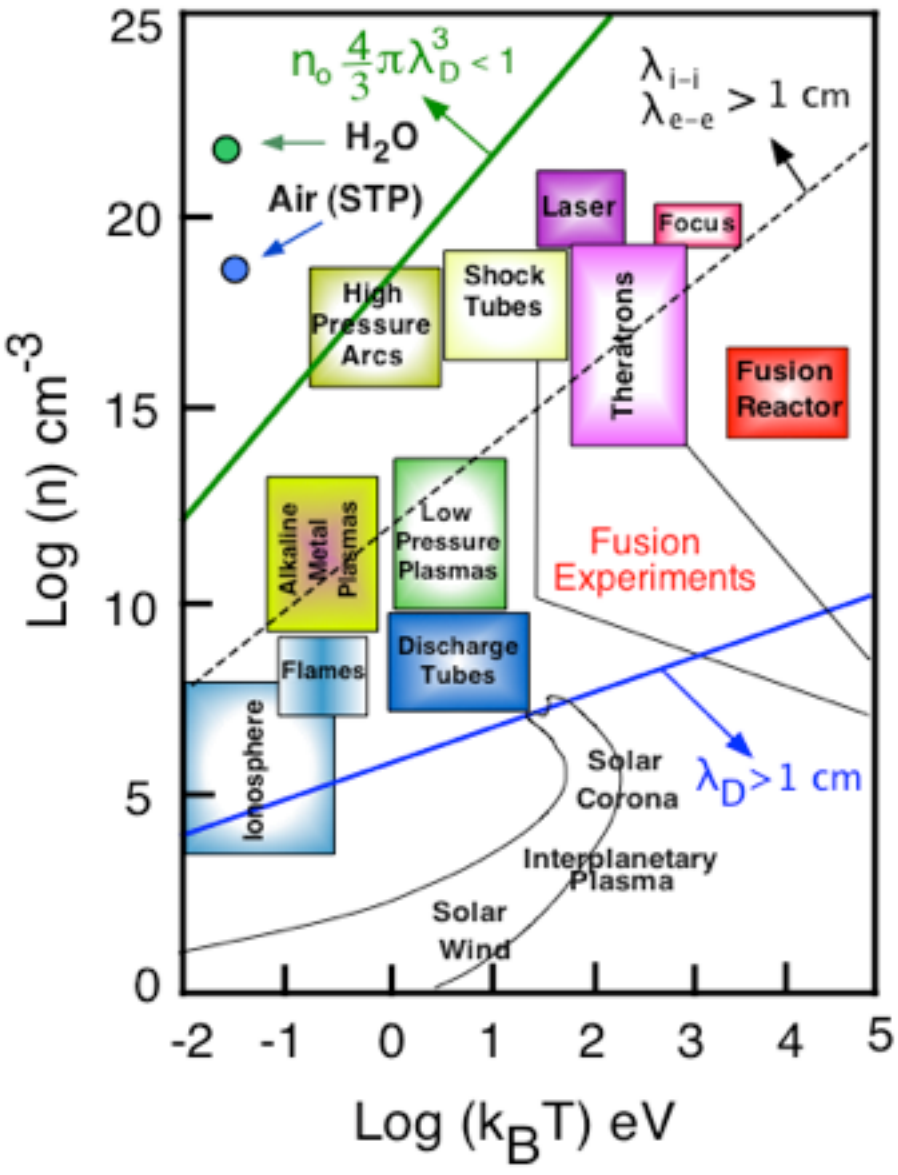
$$\frac{\partial^2 E_y}{\partial x^2} = \left(\frac{\omega_{pe}}{c}\right)^2 E_y \qquad \lambda_p = \sqrt{\frac{c}{\omega_{pe}}}$$

$$B_z(x,t) = B_o(t)\, e^{-x/\lambda_p}$$

$$E_y(x,t) = \lambda_p\, (\partial B_o/\partial t)\, e^{-x/\lambda_p}$$

The magnetic field exponentially decreases in the plasma along the so called *skin depth* of *London characteristic length*

Back to plasmas in nature and in the laboratory , ...



Plasmas are roughly classified according to the temperature and charged particles densities

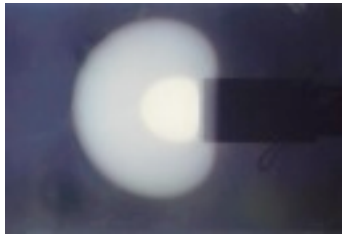
Fusion reactor: KT≈10^4 eV, n≈10^{15} cm^{-3}

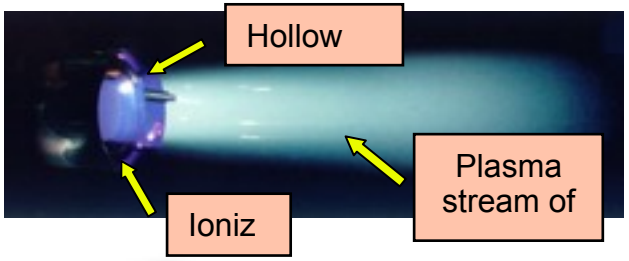
Laser plasmas: KT≈10^2 eV, n≈10^{20} cm^{-3}

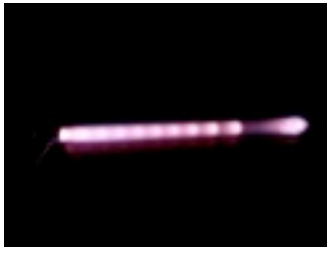
Glow discharge: KT≈1-3 eV, n≈10^8 cm^{-3}

Ionosphere: KT≈0,05 eV, n≈10^6 cm^{-3}

Water at room temperature: KT≈0,025 eV, n≈10^{22} cm^{-3}

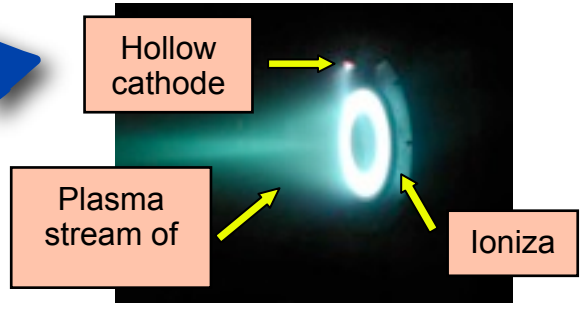






Low pressure discharges

KT≈1-3 eV, n≈10^8 cm^{-3}

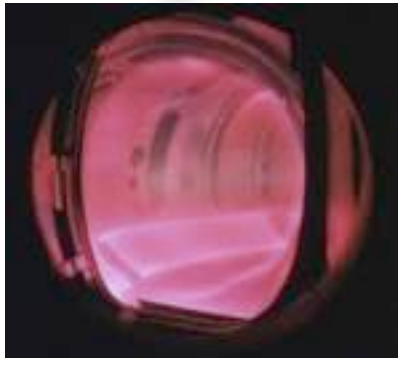


Solar corona

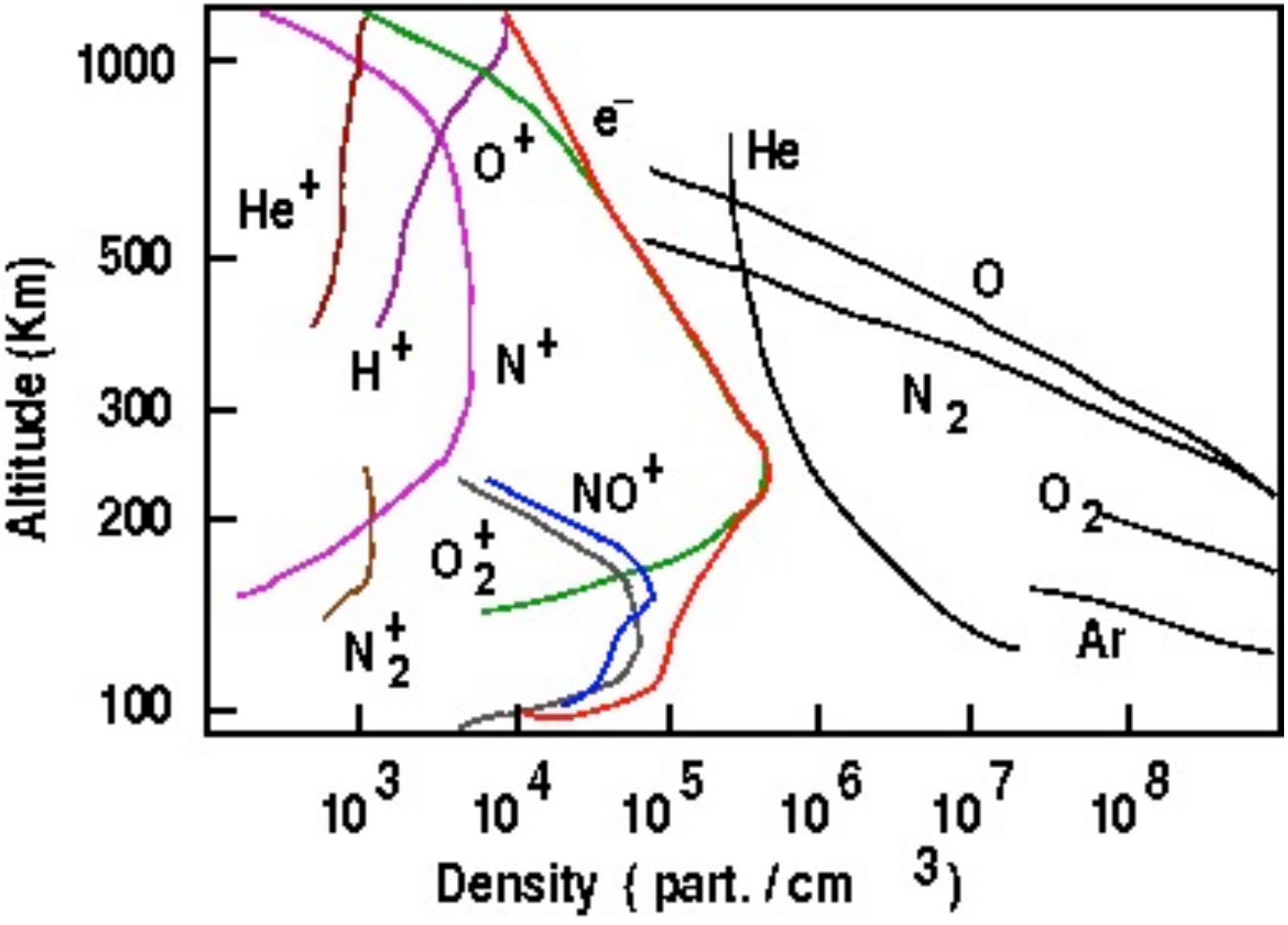
KT≈100 eV, n≈10^9 cm^{-3}

Fusion reactor

KT≈10^4 eV, n≈10^{15} cm^{-3}



The Earth ionospheric plasma, ...



The black curves are for neutral particles and the red line is the altitude dependent electron density.

The sum of all different ion densities n_i equals the electron density n_e

The ions are produced by the absorption by the neutrals of parts of the solar radiation spectrum. The maximum rate takes place at the F_1 and F_2 peaks.