The ideal Maxwellian plasma

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Plasmas are, ...

The plasma state of matter may be defined as a mixture of positively charged ions, electrons and neutral atoms which constitutes a *macroscopic electrically neutral medium* which *responds to the electric and magnetic fields in a collective mode*.

**Properties**

- Charged particles interact through *long distance electromagnetic forces* in addition to short range molecular collisions.
- The density of negative $n_e$ and positive $n_i$ charged particles are equal, so that on the average *the medium is electrically neutral* (quasineutrality).
- The *response to external perturbations is collective*, large number of charges are involved.
- We may have *multicomponent plasmas*, ions with negative charge, dusty plasmas (complex plasmas), ...etc.
For simplicity, we will limit to one component classical plasmas with single charged ions.

The particles in our plasmas $\alpha = a, i, e$ will be electrons $e$, ions $i$ and eventually neutral atoms $a$ with number densities $n_\alpha$.

The particle temperatures equivalent to the average kinetic energy of particles are usually expressed in eV and $1 \text{ eV} = 11600 \text{ K}$,

$$\frac{e}{k_B T} = \frac{1.6 \times 10^{-19}}{1.38 \times 10^{-23} \times 1, 0} = 11.594, 2 \approx 11.600 \text{ K}$$

The ion charge is $Q_i = eZ$ but in most cases we will consider only single charged ions ($Z=1$).

We will use MKSC unit system.
The equilibrium state of a neutral gas, ...

The particle energy distribution function of equilibrium system is Maxwellian with temperature $k_B T$

$$f_a(v) = \left( \frac{m_a}{2\pi k_B T} \right)^{3/2} \exp \left( -\frac{m_a v^2}{2k_B T} \right)$$

$$g_a(E) = \frac{2}{\sqrt{\pi}} \frac{\sqrt{E}}{(k_B T)^{3/2}} \exp \left( -\frac{E}{k_B T} \right)$$

The normalized probability distribution

$$\int_0^\infty g_a(E) \, dE = \int_{-\infty}^{+\infty} f_a(v) \, dv = 1$$

$$dn_a = n_{ao} f_a(v) \, dv$$

$$dn_a = n_{ao} g_a(E) \, dE$$

particles with

$$\{ (v, v + dv) \}$$

$$\{ (E, E + dE) \}$$
The macroscopic properties, ...  

The macroscopic physical properties are calculated as averages of the particle energy distribution function. The average energy per particle,  

\[
\langle E \rangle = \int_0^\infty E g_a(E) \, dE = \frac{3}{2} k_B T \quad E_i = n_{ao} \langle E \rangle = \frac{3}{2} n_{ao} k_B T
\]

The thermal speed redefines the velocity distribution function,  

\[
v_{th} = \sqrt{\frac{2 k_B T}{m_a}} \quad f_a(v) = \frac{1}{(\sqrt{\pi} v_{th})^3} \exp\left(-\frac{v^2}{v_{th}^2}\right)
\]

The velocity averages,  

\[
\langle v_x \rangle = \langle v_y \rangle = \langle v_z \rangle = 0
\]

\[
v_{ave} = \langle \sqrt{v \cdot v} \rangle = \sqrt{\frac{8k_B T}{\pi m_a}} \quad \langle |v_x| \rangle = \sqrt{\frac{2 k_B T}{\pi m_a}}
\]

define the particle flux,  

\[
\Gamma_a = \frac{1}{2} n_{ao} v_{ave} \quad J_\alpha = q_\alpha \Gamma_\alpha = q_\alpha \frac{1}{2} n_{ao} v_{ave}
\]
The equilibrium of an ideal Maxwellian plasma...

- The energy distribution function of charged particles ions and electrons is Maxwellian.

- All particle groups have the same kinetic temperature $K_B T_e = K_B T_i = K_B T$ and $K_B T$ is the temperature corresponding to the thermodynamic equilibrium.

- Plasma quasineutrality, $n_e \approx n_i$ implies that no electric fields exist in the equilibrium plasma bulk; uniform plasma potential.

- No transport; no particle currents.
The Maxwellian plasma in a external electric field ...

Background equilibrium, \( n_{eo} \approx n_{io} \)

\[
E_\alpha = \frac{m_\alpha v^2}{2} + q_\alpha \phi(x)
\]

\[
n_\alpha = n_{\alpha o} \int_0^\infty g_\alpha(E) \, dE
\]

\[
\begin{align*}
n_e(x) &= n_{eo} \exp \left( \frac{e \phi}{k_B T} \right) \\
n_i(x) &= n_{io} \exp \left( - \frac{e \phi}{k_B T} \right)
\end{align*}
\]

In cold plasma \( k_B T \approx 0 \) the charge separation is complete while in a finite temperature plasma \( k_B T \neq 0 \) the thermal and electrostatic energies compete. This permits to shield out the small amplitude fluctuations of electric and magnetic fields.
The damping and/or attenuation of small amplitude fluctuations of the equilibrium takes place over a characteristic length and time scales.

- Space fluctuations: Debye length
- Time fluctuations: Plasma frequency
- Number of charges: Plasma parameter

\[ \lambda_{De} = \sqrt{\frac{\epsilon_o k_B T}{e^2 n_{eo}}} \quad \lambda_{Di} = \sqrt{\frac{\epsilon_o k_B T}{e^2 n_{io}}} \]

\[ k_B T_e \gg k_B T_i \quad \lambda_{De} \gg \lambda_{Di} \]

\[ \omega_{pi} = \sqrt{\frac{e^2 n_{io}}{\epsilon_o m_i}} \quad \omega_{pe} = \sqrt{\frac{e^2 n_{eo}}{\epsilon_o m_e}} \quad \frac{\omega_{pi}}{\omega_{pe}} = \sqrt{\frac{m_e}{m_i}} \]
The Debye length and shielding, ...

The initial equilibrium: \( E_o = 0 \) \( n_eo \simeq n_{io} = n_o \) \( k_B T_e; k_B T_i \)

where we introduce an small perturbation in the electric charge,

\[
\begin{align*}
\delta \rho_{ext} &= q \delta(r) \\
\delta \rho_{sp} &= e [n_i(r) - n_e(r)]
\end{align*}
\]

\[
E(r) \simeq E_o + E_1(r) \quad E_1(r) = -\nabla \varphi_1(r)
\]

and \( E_1(r) \) represents the perturbed electric field.

The field \( E_1(r) \) is governed by the Poisson equation,

\[
\nabla \cdot E = \frac{\delta \rho_{ext} + \delta \rho_{sp}}{\epsilon_o}
\]

\[
\nabla \cdot E_1 = \frac{q}{\epsilon_o} \delta(r) + \frac{e}{\epsilon_o} [n_i(r) - n_e(r)]
\]

Small amplitude perturbations of the charge/electric field means that the thermal energy \(|e \varphi(r)|\) dominates over the electrostatic energy \(k_B T\)

\[
\alpha = e, i \quad \left| \frac{e \varphi_1(r)}{k_BT_\alpha} \right| \ll 1 \quad \text{we approximate} \quad n_\alpha(r) \simeq n_{\alpha o} \left( 1 \pm \frac{e \varphi_1(r)}{k_B T_\alpha} \right)
\]
\[-\nabla^2 \varphi_1(r) = \frac{1}{\varepsilon_o} \left[ \delta \rho_{ext} + \left( \frac{e^2 n_o}{k_B T_e} \varphi_1(r) \right) + \left( \frac{e^2 n_o}{k_B T_i} \varphi_1(r) \right) \right] \]
\[
\alpha = e, i
\]

that introduces the Debye lengths,
\[
\lambda_{D\alpha} = \sqrt{\frac{\varepsilon_o k_B T_\alpha}{e^2 n_o}}
\]

Setting,
\[
\frac{1}{\Lambda^2} = \frac{1}{\lambda_{De}^2} + \frac{1}{\lambda_{Di}^2} \left( \nabla^2 - \frac{1}{\Lambda^2} \right) \varphi_1(r) = -\frac{q}{\varepsilon_o} \delta(r) \quad \varphi_1(r) = \frac{q}{4\pi\varepsilon_o} \frac{e^{-r/\Lambda}}{r}
\]

The perturbations of the electric charge and/or plasma potential decay in space with an exponential rate governed by \(\lambda_D\)

The Debye length accounts for thermal effects, the size \(\sim\lambda_D\) of the perturbed region increases with \(k_B T\)

This is a linearization only valid whereas,
\[
\left| \frac{e \varphi_1(r)}{k_B T_\alpha} \right| \ll 1
\]

We deal with a non linear Poisson equation otherwise,
The plasma frequency, ...

Again the initial equilibrium:

\[ E_o = 0 \quad n_{eo} \simeq n_{io} = n_o \quad k_B T_e; k_B T_i \]

and we consider a one small perturbation of the electric charge in one dimension as in the figures,

\[ \rho = -e n_o (A \delta X) \]

\[ \int_S E_1 \cdot ds = A E_{1x} = \rho \frac{\rho}{\varepsilon_o} = -\frac{e}{\varepsilon_o} n_o (A \delta X) \]

The electric field,

\[ E_{1x} = -\frac{e}{\varepsilon_o} n_o \delta X \]

gives us an equation of motion for ions or electrons

\[ \frac{d^2}{dt^2} (\delta X) + \left[ \frac{e^2 n_o}{m_\alpha \varepsilon_o} \right] (\delta X) = 0 \]

and defines the electron/ion plasma frequencies

\[ \alpha = e, i \quad \omega_{p\alpha} = \sqrt{\frac{e^2 n_o}{m_\alpha \varepsilon_o}} \]
The plasma length and time scales, ...

The Debye length $\lambda_D$ (also called debye radius) determines the spatial scale where the small amplitude perturbations of the electric field are shielded out.

The plasmas have two time scales; ion motion $\tau_{pi} = 1/f_{pi}$ the (slow) and the electron response $\tau_{pe} = 1/f_{pe}$ (fast). The ion motion is frozen over the time scale $\tau_{pe} = 1/f_{pe}$ the electron time scale represents the faster plasma response,

$$\omega_{ie} \omega_{pe} = \sqrt{m_e m_i} = \sqrt{\frac{1}{1840 \times 40}} = \frac{1}{271} = 3.69 \times 10^{-3} \quad \text{(Argon)}$$

Alternatively, the time scale the $\tau_{pe} = 1/f_{pe}$ is proportional to the time that a thermal electron travels along a Debye length,

$$\frac{\lambda_{De}}{V_{th}} = \sqrt{\frac{\epsilon_o k_B T_e / e^2 n_o}{\sqrt{2 k_B T_e / m_e}}} = \left( \frac{\epsilon_o m_e}{2 n_o e^2} \right)^{1/2} = \frac{1}{\sqrt{2} \omega_{pe}} = \frac{\tau_{pe}}{2 \pi \sqrt{2}}$$
Approximate magnitudes in some typical plasmas ...

<table>
<thead>
<tr>
<th>Plasma</th>
<th>$n_e$ (cm$^{-3}$)</th>
<th>$K_B T_e$ (eV)</th>
<th>$\lambda_{De}$ (cm)</th>
<th>$f_{pe}$ (Hz)</th>
<th>$n_e \lambda_{De}^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interstellar gas</td>
<td>1</td>
<td>1</td>
<td>700</td>
<td>$6.0 \times 10^4$</td>
<td>$4 \times 10^8$</td>
</tr>
<tr>
<td>Solar corona</td>
<td>$10^9$</td>
<td>100</td>
<td>0.2</td>
<td>$2.0 \times 10^9$</td>
<td>$8 \times 10^6$</td>
</tr>
<tr>
<td>Solar atmosphere</td>
<td>$10^{14}$</td>
<td>1</td>
<td>$7.0 \times 10^{-5}$</td>
<td>$6.0 \times 10^{11}$</td>
<td>40</td>
</tr>
<tr>
<td>Gas discharge</td>
<td>$10^{14}$</td>
<td>$10^4$</td>
<td>$2.0 \times 10^{-3}$</td>
<td>$2.0 \times 10^{12}$</td>
<td>$6 \times 10^6$</td>
</tr>
<tr>
<td>Tokamak</td>
<td>$10^{14}$</td>
<td>$10^4$</td>
<td>$2.0 \times 10^{-3}$</td>
<td>$2.0 \times 10^{12}$</td>
<td>$6 \times 10^6$</td>
</tr>
</tbody>
</table>

The plasma and coupling parameters, ...

In order to shield out the electric field perturbations a minimum number of charges are needed within a sphere of radius $\lambda_{D_i}$. This defines the plasma parameter $N_{D_i}$ for ions and electrons,

\[ N_{De} = \frac{4\pi}{3} n_o \lambda_{De}^3 \quad k_B T_e \gg k_B T_i \quad \lambda_{De} \gg \lambda_{Di} \quad N_{De} \gg N_{Di} \]

The plasma parameter is closely related with the coupling parameter $\Gamma_C$ the ratio between thermal and electrostatic energies.

\[ \Gamma_C = \frac{r_c}{r_d} \left\{ \begin{array}{l} r_d \sim n_o^{-1/3} \quad \text{average packing of charges} \\ r_c \text{ is the closest colliding approach as in the figure} \end{array} \right. \]

\[ E(r, v_\alpha) = \frac{m_\alpha v_\alpha^2}{2} - \frac{e^2}{4\pi\epsilon_o r} \quad E(r_c, v_{th}) = 0 \]

\[ r_c = \frac{e^2}{4\pi\epsilon_o k_B T} \quad \Gamma_C = \frac{e^2 n_o^{1/3}}{4\pi \epsilon_o k_B T} \]
For the coupling parameter we also have,

\[
\Gamma_c^3 = \frac{1}{(4\pi)^3} \times \frac{1}{n_o^2} \times \left( \frac{n_o e^2}{\epsilon_o k_B T} \right)^3 = \frac{1}{(4\pi)^3} \frac{1}{n_o^2} \frac{1}{\lambda_D^6} = \frac{1}{(4\pi \times 9)} \left( \frac{3}{4\pi n_o \lambda_D^3} \right)^2
\]

and we obtain, \( \Gamma_c = \frac{1}{36\pi N_D^2} \) equivalent to, \( \Gamma_C = \frac{e^2 n_o^{1/3}}{4\pi \epsilon_o k_B T} \)

<table>
<thead>
<tr>
<th>Description</th>
<th>Plasma parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Limits</td>
<td>( \Gamma_C \gg 1 ) (( N_D \ll 1 ))</td>
</tr>
<tr>
<td>Coupling</td>
<td>Strongly coupled</td>
</tr>
<tr>
<td>Debye sphere</td>
<td>Sparsely populated</td>
</tr>
<tr>
<td>Electrostatic influence</td>
<td>Strong</td>
</tr>
<tr>
<td>Characteristic</td>
<td>Cold and dense</td>
</tr>
</tbody>
</table>
| Examples             | Laser ablation plasmas  
                         Inertial fusion experiments  
                         White dwarfs  
                         Neutron stars  
                         Ionospheric plasmas  
                         Magnetic fusion experiments  
                         Space plasmas  
                         Electric discharge plasmas  |
Magnetized plasmas, ...

The force experienced by electric charges in this course, 

\[ \mathbf{F}_\alpha = q_\alpha n_\alpha (\mathbf{E} + \mathbf{v}_\alpha \wedge \mathbf{B}) \]

this gives the cyclotron frequency, 

\[ \Omega_\alpha = \frac{q_\alpha B_\perp}{m_\alpha} \]

and using the perpendicular speed to the magnetic field lines we obtain the \textit{Larmor radius}

\[ r_{L,\alpha} = \frac{m_\alpha v_\perp}{|q_\alpha| B} \]

a new length is defined using the particle thermal speed

\[ R_{L,\alpha} = \frac{V_{th,\alpha}}{\Omega_\alpha} \]

\[ R_{L,i} = \sqrt{\frac{m_i}{m_e}} R_{L,e} \]

\[ R_{L,e} \ll R_{L,i} > L \]

\[ R_{L,e} \ll R_{L,i} < L \]
We consider the plasma electrons

\[ \mathbf{B} = B_z(x, t) \mathbf{u}_z \quad \begin{cases} \frac{\partial \mathbf{B}}{\partial t} = -\nabla \wedge \mathbf{E} \\ B_z(0, t) = B_o(t) \end{cases} \]

\[ \mathbf{E}(x, t) = E_y(x, t) \mathbf{u}_y \]

We also have,

\[ \nabla \wedge \mathbf{B} = \mu_o \mathbf{J}_e + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \simeq -e n_{eo} \mathbf{v}_e \]

and the equation of motion for electrons

\[ \frac{\partial^2 E_y}{\partial x^2} = \frac{e^2}{m_e} n_{eo} \mu_o E_y = \frac{1}{c^2} \left( \frac{e^2 n_{eo}}{\epsilon_o m_e} \right) E_y \]

\[ \frac{\partial^2 E_y}{\partial x^2} = \left( \frac{\omega_{pe}}{c} \right)^2 E_y \]

\[ B_z(x, t) = B_o(t) e^{-x/\lambda_p} \]

\[ E_y(x, t) = \lambda_p \left( \frac{\partial B_o}{\partial t} \right) e^{-x/\lambda_p} \]

The magnetic field exponentially decreases in the plasma along the so called *skin depth* of *London characteristic length*
Plasmas are roughly classified according to the temperature and charged particles densities

**Fusion reactor:** $KT \approx 10^4 \text{ eV}, \; n \approx 10^{15} \text{ cm}^{-3}$

**Laser plasmas:** $KT \approx 10^2 \text{ eV}, \; n \approx 10^{20} \text{ cm}^{-3}$

**Glow discharge:** $KT \approx 1-3 \text{ eV}, \; n \approx 10^8 \text{ cm}^{-3}$

**Ionosphere:** $KT \approx 0.05 \text{ eV}, \; n \approx 10^6 \text{ cm}^{-3}$

**Water at room temperature:** $KT \approx 0.025 \text{ eV}, \; n \approx 10^{22} \text{ cm}^{-3}$
**Low pressure discharges**

$KT \approx 1-3 \text{ eV}, \ n \approx 10^8 \text{ cm}^{-3}$

**Solar corona**

$KT \approx 100 \text{ eV}, \ n \approx 10^9 \text{ cm}^{-3}$

**Fusion reactor**

$KT \approx 10^4 \text{ eV}, \ n \approx 10^{15} \text{ cm}^{-3}$
The Earth ionospheric plasma, ...

The black curves are for neutral particles and the red line is the altitude dependent electron density.

The sum of all different ion densities $n_i$ equals the electron density $n_e$.

The ions are produced by the absorption by the neutrals of parts of the solar radiation spectrum. The maximum rate takes place at the $F_1$ and $F_2$ peaks.