Introductory kinetic theory of gases and plasmas

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Resumen

The basic principles of kinetic theory for the equilibrium state of a neutral and/or partially ionized gas are briefly reviewed as a complementary material for the students of the *Space Environment* course of the *Máster Universitario en Ingeniería Aeronáutica*

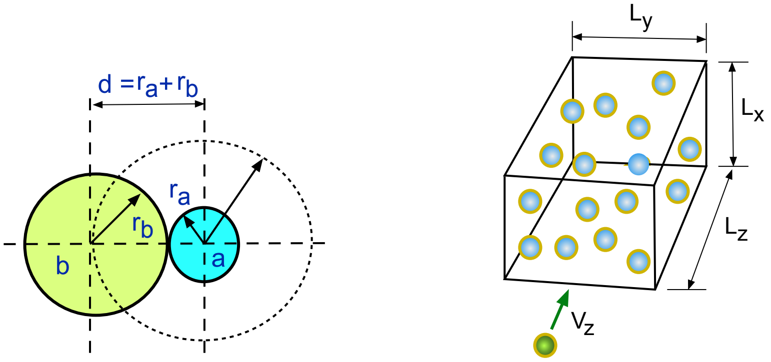
1. Neutral gases and plasmas

In classical kinetic theory particles (neutral atoms, electrons and ions) are represented by a huge number of point-like symmetrical force centers that move randomly exchanging energy and momentum by collisions. The ionization degree $\alpha_g = n_i/n_a$ is the ratio between the number density of neutral atoms n_a and the number density of electrons and ions $n_i \simeq n_e$. In the equilibrium state of partially ionized gases where $\alpha_g > 0$ the densities of positive and negative charged particles are approximately equal and thus no electric field exists in the medium.

In non-ionized gases, atoms and molecules are electrically neutral and their collisional interaction requires of a close approach typically to distances comparable to their size, where intense electric fields develops. Between two successive collisions they move along a straight line (no external force is applied to the system so far) and when the gas pressure is low enough collisions are *binary* encounters since it is unlikely three-particle collisions occur. Therefore, *short-range interactions* are dominant and *elastic collisions* between molecules are responsible for energy and momentum transport in neutral gases. The *inelastic collisions* where the internal state of atoms change (for example, a ionization event), requires higher energy of colliding particles.

When a gas is partially ionized, electric charges interact through electromagnetic forces. In addition to *short-range* collisions with neutral atoms, charged particles interact on the microscopic scale via *long-range* Coulomb forces. The motion of charged particle is affected by ions and electrons located at distant points, and therefore ionized gases respond in a *collective* mode to external perturbations. The motion of electrons, ions and neutral atoms is coupled by both short-range collisions and long-range Coulomb interactions.

Long and short-range collisions give raise to a number of physical processes (or reactions) at the microscopic level, such as ionization, light emission, etc. These are usually called *elementary processes* and are *elastic* when the internal state of colliding particles remains unchanged and *inelastic* otherwise. The collision is said *superelastic* when the energy of internal states of molecules is transformed into kinetic energy of particles. The rate at which energy or momentum is transferred between the plasma particles depends on the specific properties of the molecular encounter.



(a) The dotted circle indicates the total cross section σ_{ab} of colliding particles.

(b) The green test particle moves into the volume occupied by targets.

Figura 1: Schemes to illustrate the concept of total cross section of colliding particles.

1.1. Particle collisions in neutral gases and plasmas

We can formulate a simple model for short-range molecular encounters of neutral molecules with other atoms, ions and electrons. Since the collision occurs when two particles with radii r_a and r_b come into close contact, we consider it takes place when are separated a distance $d \leq (r_a + r_b)$ as shows the figure (1a). This defines the dotted *interaction surface* $\sigma_{ab} = \pi\,(r_a + r_b)^2$ denominated *total cross section.*

The practical use of this concept is illustrated in the figure 1b where a green particle moves towards the volume $V = L_x\, L_y\, L_z$ where number $N_b = n_b \times V$ of blue targets are randomly distributed. The motion of the incoming particle is blocked when the interaction surfaces $\sigma_{ab} \times N_b$ of blue atoms cover the cross sectional area of $L_x\, L_y = \sigma_{ab} \times\, n_b \times (L_x\, L_y\, L_z)$. Thus, the maximum depth L_z that the green atom can reach without colliding is,

$$\lambda_c = 1/(n_b\, \sigma_{ab}) \tag{1}$$

The length λ_c is called *mean free path* and represents the distance the incoming particle travels between two successive collisions. The time τ_c elapsed between these encounters is,

$$\tau_c = \frac{\lambda_c}{v_z} = \frac{1}{\sigma_{ab}\, n_b\, v_z}$$

where the velocity v_z is assumed constant. Equivalently,

$$\nu_c = \frac{1}{\tau_c} = \sigma_{ab}\, n_b\, v_z = n_b\, K \tag{2}$$

is the *collision frequency* that characterizes the number of collision events per test particle by time unit. The *reaction rate* constant is $K = \sigma_{ab}\, v_z$ and if we multiply by n_a,

$$\nu_{ab} = n_a\, n_b\, \sigma_{ab}\, v_z \tag{3}$$

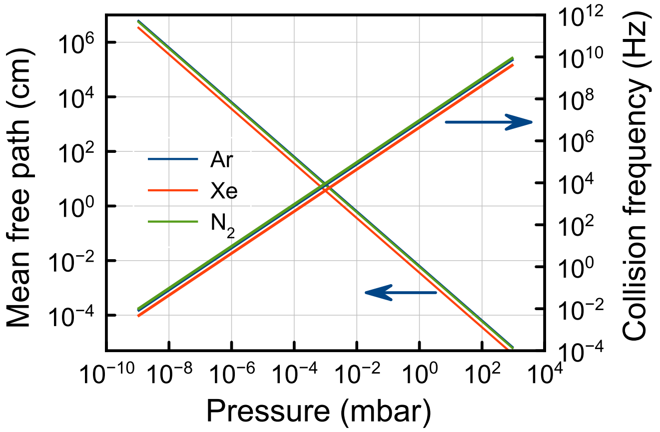


Figura 3: The mean free path λ_c (left) and the collision frequency ν_c (right) for the gas pressure p_a calculated with the values of table (1).

is the *total collision frequency*, which represents the number of collisions between the green and blue particles.

Both, the mean free path and collision frequency are probabilistic concepts used to characterize the short-range molecular collisions. The figure 2 shows a beam of green particles moving with velocity v_z into a slab thickness dz occupied by targets. The probability that an incoming particle collides is the ratio between the area $A_t = \sigma_{ab} \times n_b \times (L_x\, L_y\, dz)$ of targets and the cross-sectional area of the slab,

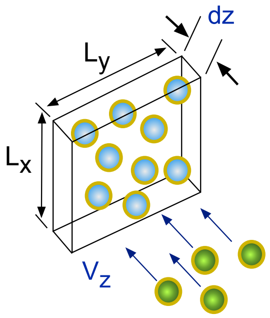


Figura 2: Target particles within $L_x \times L_y \times dz$.

$$P = \frac{A_t}{A} = \frac{\sigma_{ab}\, n_b\, (L_x\, L_y\, dz)}{L_x\, L_y} = \sigma_{ab}\, n_b\, dz$$

Along the Z axis the decrement in the initial flow of incoming green particles $\Gamma_{ao} = n_{ao}\, v_z$ is,

$$d\Gamma = -\Gamma(z) \times (\sigma_{ab}\, n_a(z)\, dz) = -\frac{\Gamma(z)}{\lambda_c}\, dz$$

where $n_a(z)$ is the density change along the slab. Integration gives,

$$\Gamma(z) = \Gamma_o\, e^{-z/\lambda_c}$$

The initial flux Γ_o exponentially decreases at the rate $\lambda_c = 1/n_b\, \sigma_{ab}$ as green particles move through the targets. We can conclude the total cross section σ_{ab} characterizes the attenuation of the incoming particle flux as they react or collide with the random background of the target blue atoms.

For gases at moderate pressures and temperatures we can replace in the previous expressions (1) and (2) the target neutral atom density $n_t = N/V$ using the ideal gas law $p_a\, V = N\, k_B T$ and the mean free path and collision frequency become,

$$\lambda_c = \left(\frac{k_B}{\sigma_{ab}}\right) \times \frac{T}{p_a} \quad \text{and,} \quad \nu_c = \left(\frac{\sigma_{ab}}{k_B}\, v_z\right) \times \frac{p_a}{T}$$

Cuadro 1: The $\lambda_c \times p$ values at $0\,^\circ\mathrm{C}$ for selected gases.

Gas	Chemical Symbol	$\lambda_c \times p$ m · mbar	$\lambda_c \times p$ m · Pa
Hydrogen	H_2	$11{,}5 \cdot 10^{-5}$	$11{,}5 \cdot 10^{-3}$
Nitrogen	N_2	$5{,}9 \cdot 10^{-5}$	$5{,}9 \cdot 10^{-3}$
Helium	He	$17{,}5 \cdot 10^{-5}$	$17{,}5 \cdot 10^{-3}$
Argon	Ar	$6{,}4 \cdot 10^{-5}$	$6{,}4 \cdot 10^{-3}$
Xenon	Xe	$3{,}6 \cdot 10^{-5}$	$3{,}6 \cdot 10^{-3}$
Air		$6{,}7 \cdot 10^{-5}$	$6{,}7 \cdot 10^{-3}$

Here v_z is the characteristic velocity of particles and we will see that for a gas in thermal equilibrium $v_z \sim \bar{v} = (8k_BT/\pi m)^{1/2}$ is the average velocity of molecules. For the 273,15 K ($0\,^\circ\mathrm{C}$) temperature we obtain 506 m/s for argon atoms, 280 m/s for Xenon and 603 m/s N_2 molecules.

For constant temperature, above equations show that $\lambda_c \times p \propto C(T)$ is a constant and its experimental values for 273,15 K ($0\,^\circ\mathrm{C}$) are in table (1), also the figure (3) shows the corresponding mean free paths and collision frequencies. For the pressure of p = 1 mbar (100 Pa) the mean free path in Argon is $\lambda_c = 6,4 \cdot 10^{-5}$ m that increases to λ_c = 0,064 m for $p = 10^{-3}$ mbar (0.1 Pa). The corresponding collision frequencies are $\nu_c = 9,4 \cdot 10^7$ s^{-1} (9,4 MHz) and $\nu_c = 9,4 \cdot 10^3$ s^{-1} (9,4 KHz).

In neutral gases the values of λ_c decrease as the pressure increments whereas ν_c grow and both are related with the total cross section σ_t for short-range elastic collision between gas molecules. Then, λ_c and $\tau_c = 1/\nu_c$ are the characteristic length and time scale which depend on the gas pressure and also on its temperature, as it increases the mean kinetic energy of molecules which also affects to the collision rate.

When the gas is partially ionized ($\alpha_g > 0$) the long range electromagnetic forces introduce additional mean free path ($\lambda_{ee}, \lambda_{ei}, \lambda_{ii}$) and time scales ($\tau_{ee}, \tau_{ei}, \tau_{ii}$) for Coulomb collisions between charged particles. These length and time scales provide us with a criterion to consider a ionized gas as a plasma.

Besides an appreciable ionization degree $\alpha_g > 0$ the mean free path λ_c for the dominant collisional process needs to be much smaller than the macroscopic dimensions of the system $L \gg \lambda_c$ and also the time scale $T \gg \tau_c = 1/\nu_c$ considered much longer than the characteristic time associated to the collision frequency. Hence, a huge number of molecular encounters take place within the volume L_s^3 during the characteristic time τ_c in these conditions. Then we can say,

> The *plasma state of condensed matter* is a mixture of positively charged particles, electrons and neutral atoms that constitutes a *macroscopic electrically neutral medium that responds to the electric and magnetic fields in a collective mode.*

However, a number of different collisional processes (light emission, excitation, ionization, ...) are simultaneously present in ionized gases at the atomic and molecular level. Strictly speaking, the thermodynamic equilibrium of a plasma requires the detailed balance of each reaction or elementary process by its reverse collision. Such situation is seldom found in nature and in the laboratory and most plasmas are physical systems far from equilibrium since they need an external source of energy to exist, unlike condensed matter in liquid, gas or solid state.

In the following we will limit ourselves to classical plasmas composed by neutral atoms, electrons and single charged ions for simplicity. Within the time scale of interest electromagnetic interaction will be regarded as instantaneous, so relativistic effects will not be considered. For the time evolution of electromagnetic fields we will make use of the Maxwell equations in a vacuum,

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_c}{\epsilon_o} \qquad \text{Gauss's law} \tag{4a}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \text{Faraday induction law} \tag{4b}$$

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \text{Gauss's law form magnetism} \tag{4c}$$

$$\nabla \times \boldsymbol{B} = \mu_o \left(\boldsymbol{J} + \epsilon_o \frac{\partial \boldsymbol{E}}{\partial t} \right) \qquad \text{Ampere-Maxwell law} \tag{4d}$$

Here, $c^2 = 1/\mu_o \, \epsilon_o$ is the speed of light, the electric field is $\boldsymbol{E}(\boldsymbol{r}, t) = -\nabla \, \phi(\boldsymbol{r}, t)$, where $\phi(\boldsymbol{r}, t)$ is the electric potential and $\boldsymbol{B}(\boldsymbol{r}, t)$ the magnetic induction. The electric charge density and electric current are respectively $\rho_c(\boldsymbol{r}, t)$ and $\boldsymbol{J}(\boldsymbol{r}, t)$. These can be related to the plasma parameters as the negative charge density is $\rho_e = -e \, n_e$ and the electron current $\boldsymbol{J}_e = -e \, \boldsymbol{\Gamma}_e$ where the negative charge flow of charge is $\boldsymbol{\Gamma}_e = -e \, n_e \boldsymbol{u}_e$ and $\boldsymbol{u}_e$ the average electron velocity.

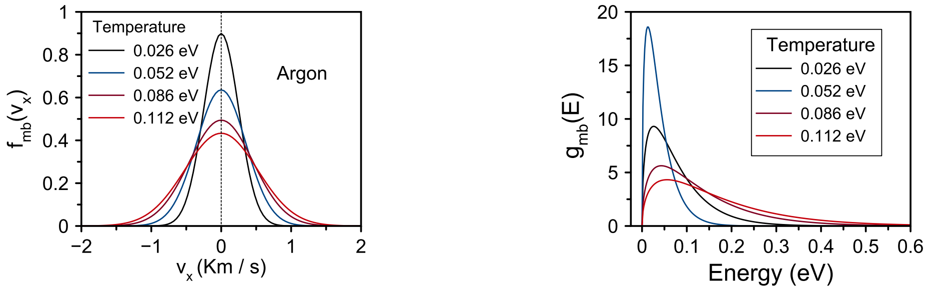
2. Basic kinetic theory

Next, we introduce the simplest kinetic description of the equilibrium state of a neutral gas composed of a huge number of interacting particles. These basic principles can be extended to a plasma of electrons, ions and neutral atoms. Atoms and molecules are considered as points with mass m that move randomly within the system. The macroscopic characteristic length L of the system is much longer than the mean free path for the dominant collision process is $L \gg \lambda_c$. Typical distances that particles travel are much longer than the dimensions of atoms and molecules. Furthermore, within the macroscopic time scale $T \gg \tau_c = 1/\nu_c$ collisions between atoms and molecules are frequent.

Kinetic theory make use of a probabilistic approach to describe the equilibrium and non-equilibrium physical systems of interacting particles. where the macroscopic magnitudes such as temperature or pressure are calculated as statistical averages. Specific properties of collisional processes, as the cross sections, determine transport properties as thermal conductivity or viscosity.

In the time-independent equilibrium state of a gas pressure, temperature and atom density n_o are uniform and its particles move randomly. The *velocity distribution function* $f(\boldsymbol{v})$ describes the velocity spectrum of atoms and expresses the probability of finding a molecule with a velocity component between $\boldsymbol{v}$ and $\boldsymbol{v} + d\boldsymbol{v}$ is $dP = f(\boldsymbol{v}) \, d\boldsymbol{v}$ (equivalently, $dP = f(\boldsymbol{v}) \, dv_x \, dv_y \, dv_z$). The corresponding number dn of particles is $dn = n_o \, f(\boldsymbol{v}) \, d\boldsymbol{v}$ (or also $dn = n_o \, f(\boldsymbol{v}) \, dv_x \, dv_y \, dv_z$).

The number of particles in the system is a constant as well as its total energy, which is the sum of the energies of all particles, therefore, the velocity distribution function (equivalently, the probability) is bounded,



(a) In one dimension for Argon as a function of v_x.

(b) In 3D as a function of the energy $E = m\boldsymbol{v}^2/2$.

Figura 4: The Maxwell-Boltzmann distribution function (6) for different temperatures $k_B T$ expressed in electron-volts.

$$\int_{-\infty}^{\infty} f(\boldsymbol{v})\, d\boldsymbol{v} = \int_{-\infty}^{\infty} f(\boldsymbol{v})\, dv_x\, dv_y\, dv_z = A$$

where A is a constant.

In the equilibrium state, the function $f(\boldsymbol{v})$ is the same for all points within the gas and, since there is not a preferred direction for the motion of particles, each velocity component can be treated independently. We have,

$$dP = f(\boldsymbol{v})\, d^3 v = (f(v_x)\, dv_x) \times (f(v_y)\, dv_y) \times (f(v_z)\, dv_z) = f(v_x)\, f(v_y)\, f(v_z)\, dv_x\, dv_y\, dv_z$$

and since the direction of molecule motion is random we have $f(\boldsymbol{v}) = f(-\boldsymbol{v})$ and then,

$$f(\boldsymbol{v}) = f(v_x^2 + v_y^2 + v_z^2) = f(v_x^2)\, f(v_y^2)\, f(v_z^2)$$

and the only function that satisfies $f(a, b, c) = f(a)\, f(b)\, f(c)$ is a power of the speed components. We can write,

$$f(v_x) = A\, e^{-v_x^2/B} \tag{5}$$

and similar expressions hold for v_y and v_y. The constant $B > 0$ is positive to keep the velocity distribution function bounded. We can introduce the dimensionless ratio between the kinetic energy of molecules and the thermal energy of the gas,

$$\frac{v_x^2 + v_y^2 + v_z^2}{B} = \frac{(m\, \boldsymbol{v}^2/2)}{(k_B T)}$$

and the integration gives $A = (2\,\pi\, k_B\, T/m)^{3/2}$. Finally, the equilibrium velocity distribution of particles with mass m at the temperature T is given by the *Maxwell-Boltzmann velocity distribution function*,

$$f_{mb}(\boldsymbol{v}) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{m\, \boldsymbol{v}^2}{2 k_B\, T}\right) \tag{6}$$

Alternatively, $dP = f_{mb}(\boldsymbol{v})\,d\boldsymbol{v}$ can be seen as the probability of finding one molecule with velocity between $\boldsymbol{v}$ and $\boldsymbol{v} + d\boldsymbol{v}$ and $dn = n_o\,dP$ is the average number of these particles. The Maxwell-Boltzmann distribution in one dimension is represented in figure (4a) for different temperatures expressed in electron-volts (1 eV = 11600 K), it has a sharp maximum for $v_x = 0$ and $f(v_x)$ decreases with the particle speed. The temperature is a measure of its thickness and figure (4a) shows the distribution becomes sharper as $k_B T$ decreases.

We can express the Maxwell-Boltzmann distribution as a function of the energy $E = m\,v^2/2$ because equation (6) is only a function of $\boldsymbol{v} \cdot \boldsymbol{v} = v^2$, so using spherical coordinates $d^3v = v^2\ \sin\theta d\theta\, d\phi\, dv$ we have,

$$dP = f_{mb}(\boldsymbol{v})\,d\boldsymbol{v} = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp{(-\frac{m\,v^2}{2\,k_B T})}\, v^2\ \sin\theta\ d\theta\, d\phi\, dv$$

Integration over ϕ and θ gives,

$$f_{mb}(v) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp{(-\frac{m\,v^2}{2\,k_B T})}\,(4\pi v^2)$$

and we can write the probability as $dP = g_{mb}(E)dE$ where,

$$g_{mb}(E) = \frac{2}{\sqrt{\pi}}\,\frac{\sqrt{E}}{(k_B T)^{3/2}}\,e^{-E/k_B T} \tag{7}$$

as a function of the particle energy.

The equation (7) is represented in figure (4b) for different temperatures and it has a sharp maximum for $E = k_B T/2$. In this case $dP = g_{mb}(E)\,dE$ represents the probability of finding a particle with kinetic energy between E and $E + dE$ and then,

$$\int_0^\infty g_{mb}(E)\,dE = 1$$

Figure (4b) also shows the energy distribution $g_{mb}(E)$ becomes sharper as the temperature $k_B T$ decreases as in equation (6).

3. Averaging over distributions

The velocity distribution functions (6) and (7) permit to calculate the average of a function $H(\boldsymbol{v})$ related with a physical magnitude as,

$$\langle\, H(\boldsymbol{v})\,\rangle = \int_{-\infty}^{+\infty} H(\boldsymbol{v})\, f_{mb}(\boldsymbol{v})\, d^3v \ \ \text{or also,}\ \ \langle\, H(E)\,\rangle = \int_0^\infty H(E)\, g_{mb}(E)\, dE \tag{8}$$

where $dP = f_{mb}(\boldsymbol{v})\,d\boldsymbol{v}$ or $dP = g_{mb}(E)\,dE$ are the probability densities. The simplest averages,

$$\langle\, n\,\rangle = n_o \int_{-\infty}^{+\infty} f_{mb}(\boldsymbol{v})\, d^3v = n_o \ \ \text{and,}\ \ \langle\, n\,\rangle = n_o \int_0^\infty g_{mb}(E)\, dE = n_o$$

give the particle number density n_o and,

$$N_a = \int_{-\infty}^{+\infty} n_o\, d^3r = n_o \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{mb}(\boldsymbol{v})\, d^3v\, d^3r$$

represents the total number of particles N in the system.

Using the Maxwell-Boltzmann distribution function (6) or (7) we recover the properties of an ideal gas. The equilibrium gas temperature is related with the average kinetic energy $e_i = \langle m\,\boldsymbol{v}^2/2\rangle$ per particle

$$e_i = \langle\, m\,\boldsymbol{v}^2/2\,\rangle = \int_{-\infty}^{+\infty} f_{mb}(v)\,(\frac{mv^2}{2})\,d^3v = \frac{3}{2}\,k_BT \tag{9}$$

and equivalently,

$$e_i = \int_0^{\infty} E\,g_{mb}(E)\,dE = \frac{3}{2}\,k_BT$$

Therefore, the internal energy by volume unit of this ideal gas is $U_i = n_o\,e_i$ since the Maxwell-Boltzmann distribution describes its equilibrium state.

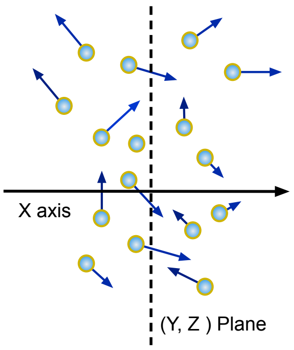


Figura 5: Random motion of particles crossing the (Y, Z) plane.

Different characteristic velocities are defined as averages over the distribution function, as the *root mean square speed* (rms) $v_R = \sqrt{\langle\, v^2\,\rangle}$ as,

$$v_R^2 = \langle v^2\rangle = \int_{-\infty}^{+\infty} v^2\,f_{mb}(\boldsymbol{v})\,d^3v$$

$$v_R^2 = \int_0^{+\infty} f_{mb}(\boldsymbol{v})\,v^2\,(4\pi\,v^2\,)\,dv = \left(\frac{3\,k_BT}{m}\right) \tag{10}$$

which is related with the mean kinetic energy since $e_i = m\,v_R^2/2$. The mean *mean thermal speed* defined as $\bar{v} = \langle\,|\boldsymbol{v}|\,\rangle = \langle\,v\,\rangle$ is,

$$\bar{v} = \langle v\rangle = \int_{-\infty}^{+\infty} v\,f_{mb}(\boldsymbol{v})\,d^3v = \int_0^{+\infty} f_{mb}(\boldsymbol{v})\,v\,(4\pi\,v^2\,)\,dv = \left(\frac{8\,k_BT}{\pi\,m}\right)^{1/2} \tag{11}$$

The *most probable velocity* for a particle is called the *thermal speed*,

$$v_{th} = \sqrt{\frac{2\,k_B\,T}{m}} \tag{12}$$

and the Maxwell-Boltzmann velocity distribution (6) can be also cast as,

$$\bar{v} = \langle v\rangle = \int_{-\infty}^{+\infty} v\,f_{mb}(\boldsymbol{v})\,d^3v = \int_0^{+\infty} f_{mb}(\boldsymbol{v})\,v\,(4\pi\,v^2\,)\,dv = \left(\frac{8\,k_BT}{\pi\,m}\right)^{1/2} \tag{13}$$

$$f_{mb}(\boldsymbol{v}) = \frac{1}{(v_{th}\,\sqrt{\pi})^3}\,\exp(-\boldsymbol{v}^2/v_{th}^2)$$

The *mean thermal speed* $\bar{v} = \langle\,|\boldsymbol{v}|\,\rangle = \langle\,v\,\rangle$ is also defined using (6) as,

In the general case the drift velocities that characterize the macroscopic motion of gas molecules can be also defined from the averages (8) using the *non-equilibrium* velocity distribution function different from (6). Strictly speaking, the Maxwell-Boltzmann distribution $f_{mb}(\boldsymbol{v})$ (or $g_{mb}(E)$) only applies to the thermodynamic equilibrium state of a low-pressure ideal gas where there is no macroscopic transport of mass, kinetic energy or momentum.

Since molecules are always in motion, the average energy $\langle\, m\,\boldsymbol{v}^2/2 \,\rangle$ is positive (see equation 10). However, the average velocity along a fixed direction $\langle\, v_x \,\rangle$ is null,

$$\langle\, v_x \,\rangle = \int_{-\infty}^{+\infty} v_x\, f_{mb}(\boldsymbol{v})\, d^3v = \int_{-\infty}^{+\infty} v_x\, f_{mb}(v_x)\, dv_x = 0$$

so we have the same average number of molecules with velocity $v_x > 0$ and $v_x < 0$. That is, if we mentally drawn a plane in a gas, the mean number of molecules crossing from left to right is the same as that those moving in the opposite direction, as shows figure (5). Therefore, $\langle\, v_x \,\rangle = \langle\, v_y \,\rangle = \langle\, v_z \,\rangle = 0$. We recover the fact that there is not a privileged direction for the motion of molecules in the thermal equilibrium state of an ideal gas.

However, we can evaluate the mean velocity of particles that cross the plane of figure (5) from left to right ($v_x > 0$) as,

$$\overline{v_x} = \int_0^{\infty} v_x\, dP = \int_0^{\infty} v_x\, f(v_x)\, dv_x = \left(\frac{k_B T}{2\,\pi\, m}\right)^{1/2}$$

Then, the particle flux,

$$\Gamma_x = n_o\, \overline{v}_x = \frac{1}{2}\, n\, \langle |v_x| \rangle = \frac{1}{4}\, n_o\, \bar{v}$$

represents the mean number of particles crossing the plane of figure (5) moving with $v_x > 0$. Since the X axis points along an arbitrary direction we can write $\boldsymbol{\Gamma} = n_o\, \overline{v}_n\, \boldsymbol{n}$ where $\boldsymbol{n}$ is the unit vector along a fixed direction. When particles have a charge q the vector $\boldsymbol{J}_q = q\, \boldsymbol{\Gamma} = q\, \overline{v}_n\, \boldsymbol{n}$ gives the electric current density crossing the plane; that is, the average electric charge passing through by time and surface units.

The Maxwell-Boltzmann distribution functions (6) and (7) also apply to the equilibrium state of a mixture of non-chemically reactive gases. The short-range collisions transfer energy and momentum between different molecular species and in the equilibrium state their average kinetic energy becomes equal and the mixture reaches a common temperature. However, their characteristic velocities (12) or (13) will be different, in accordance to the mass of each particle.

4. The neutral gas in a force field

External forces acting on a neutral gas, are not included in the Maxwell-Boltzmann distribution functions (6) and (7). When particles of mass m are subjected to the force $\boldsymbol{F}(\boldsymbol{r}) = -\nabla\, V$ where $V(\boldsymbol{r})$ is the potential (for example, the gravitational potential) we can write,

$$E = E_k + V(\boldsymbol{r}) = \frac{m\,\boldsymbol{v}^2}{2} + V(\boldsymbol{r})$$

Using the equation (7) the number of molecules per unit volume with kinetic energies within the interval E_k and $E_k + dE_k$ is,

$$n(E_k)\, dE_k = n_o\, \frac{2}{\sqrt{\pi}}\, \frac{\sqrt{E_k}}{(k_B T)^{3/2}}\, \exp\left(-\frac{E_k + V(\boldsymbol{r})}{k_B\, T}\right)$$

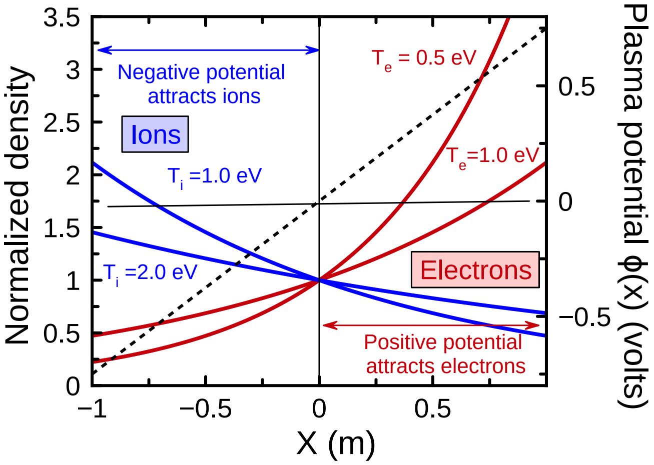


Figura 6: The electron $n_e(\phi)$ (red curve) and ion $n_i(\phi)$ (blue curve) densities of equations (14). represented as a function of the X coordinate for a linear potential profile $\phi(x) \simeq \boldsymbol{E}_o\, x$ (dotted curve, right axis).

and a similar expression can be derived from equation (6). The integration over E_k gives,

$$n(\boldsymbol{r}) = n_o\, e^{-V(\boldsymbol{r})/k_B\,T}$$

and the so-called *Boltzmann factor* $e^{-V(\boldsymbol{r})/k_B\,T}$ introduces a change in the spatial distribution of particle density. Specifically, for the Earth's gravitational field in one dimension we have $V(h) = m\, g_o\, h$ where g_o is the standard acceleration and,

$$n(h) = n_o\, \exp(-m\, g_o\, h/k_B\, T)$$

is the decreasing concentration of particles with the altitude h and $n_0 = n_o$ the value at the ground level. This change is negligible on the laboratory scale where the particle density n_o can be considered uniform but can be important for gases on Earth or planetary length scales.

The energy of charged particles subjected to a external electric field $\boldsymbol{E} = -\nabla\phi$ is,

$$E = E_k + V(\boldsymbol{r}) = \frac{m\, \boldsymbol{v}^2}{2} \pm e\, \phi(\boldsymbol{r})$$

where $+e$ stand for ions and $-e$ for electrons. Following the previous arguments we obtain for the electric charges,

$$n_i(\phi) = n_o\, e^{-\phi(\boldsymbol{r})/k_B\,T} \qquad \text{and for the electrons,} \qquad n_e(\phi) = n_o\, e^{\phi(\boldsymbol{r})/k_B\,T} \tag{14}$$

that are represented in figure (6) for a constant electric field.

In the equilibrium when no electric field $\phi = 0$ is applied $n_e(0) = n_i(0) = n_o$ the positive and negative charged particle densities are equal and we recover the *quasineutrality* of a plasma. Electrons concentrate at the high potential side ($\phi(\boldsymbol{r}) > 0$) whereas ion density increases at the negative potential side ($\phi(\boldsymbol{r}) < 0$).

However, figure (6) shows that this separation of positive and negative charges is not perfect since it also depends on the temperature of the system, which is related with the average kinetic energy per particle (see equation 9). Electrons (ions) in the high energy tail of the Maxwell-Bolzmann distribution in figures (4a) and (4b) can penetrate on the negative (positive) potential side. For this reason figure (6) shows the electron (ion) density n_e (n_i) is not null for negative (positive) values of the electric potential.

This statistical effect in *finite temperature plasmas* where $k_B T > 0$ is not possible in the limit of a *cold* electron (ion) populations where $k_B T \to 0$ since the distributions of figure 4 become very narrow having a much reduced number of particles in the high energy tail. For low $k_B T$ values, only a much limited number of electric charges have energy enough to overcome the potential barrier and charge separation is more effective.