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An Introduction to Plasma Physics and its Space Applications, Volume 2

Basic equations and applications

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Chapter 3

Hydrodynamic description of plasmas

The hydrodynamic equations for a plasma can be derived from the Boltzmann equations for ions, electrons and neutral atoms by means of statistical averages. Each plasma species is described by an independent transport equation coupled to the others by macroscopic *friction* terms that account for their collisional interactions. This the complete hydrodynamic plasma equations are using physical assumptions as in the *cold* and *warm* plasma limits.

We have seen in chapter 2 that the mathematical formulation for collision operators $C(f_\alpha)$ for plasma species ($\alpha = e, i, a$) are cumbersome. The derivation of general expressions of non-equilibrium probability distribution functions $f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)$ as solutions of the kinetic equation (2.10) is a mathematical challenge. However, we can derive simpler and more intuitive fluid equations to describe the macroscopic transport of mass, energy and momentum in a plasma, which are valid under certain approximations.

The neutral gases can be seen as macroscopic continuous fluid when the dimensionless Knudsen number,

$$Kn = \frac{\lambda_c}{L_s} = \frac{k_B\, T}{\sqrt{2}\,\pi\,\sigma_c^2\, p_a\, L_s} \ll 1$$

is small, as shown in the scheme of figure 1.1. This approximation is valid, so far the characteristic dimension $L_s \gg \lambda_c$ of the physical system is much larger than the collisional mean free path λ_c for short-range molecular interactions[1]. The

[1] See equation volume 1, equation (2.2), box 2.2 and section 3.4.

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macroscopic transport equations are simpler than kinetic theory and make use of a small number of physical parameters, such as the fluid velocity $\boldsymbol{u}(\boldsymbol{r}, t)$, temperature $T(\boldsymbol{r}, t)$ and the particle density $n(\boldsymbol{r}, t)$ or, equivalently, the pressure $p(\boldsymbol{r}, t)$. These magnitudes are statistical averages over a large number of particles[2] that change with position $\boldsymbol{r}$ and also evolve over time in the general case.

These assumptions can be extended to plasma theory and therefore, plasmas can be considered as *macroscopic continuous media* under certain conditions. The ions, electrons and neutral atoms can be regarded as three mutually inter-penetrating fluids, whose motions are coupled by both, short-range collisions and long-range electromagnetic interactions. In hydrodynamic models are introduced the densities $n_\alpha(\boldsymbol{r}, t)$, fluid velocities $\boldsymbol{u}_\alpha(\boldsymbol{r}, t)$ and local temperatures $T_\alpha(\boldsymbol{r}, t)$ of plasma species ($\alpha = a, e, i$), similarly to transport equations of neutral fluids and gases.

In addition to the mean free path λ_c for the dominant collisional process, we need also to consider the Debye length scale[3] for the local electric field shielding. Furthermore, the number of charges $N_{ch} \sim n_\alpha \lambda_D^3 \gg 1$ needs to be large enough to account for the Debye shielding.

The hydrodynamic approximation for a plasma is appealing when a large number of charged particles are into a small cube with lateral size $l_c \gg \lambda_c$ much smaller than the relevant macroscopic dimension of the system $L_s \gg l_c \gtrsim \lambda_D$, but longer than the Debye length. We have $n_\alpha l_c^3 \gg 1$ and also $n_\alpha \lambda_D^3 \gg 1$, in this case the averaged physical magnitudes such as the electric charge density $\rho_c(\boldsymbol{r}, t)$ or current density $\boldsymbol{J}_c(\boldsymbol{r}, t)$ are statistically valid concepts.

Fluid models basically apply to weakly coupled LTE plasmas where particle collisions along $l_c \ll L_s$ relax the random fluctuations of energy $\delta E_\alpha \sim \delta T_\alpha \ll T_\alpha$ and particle densities $\delta n_\alpha \ll n_\alpha$ over a time scale $\tau \ll L_s/|\boldsymbol{u}_\alpha|$ much shorter than the macroscopic plasma motion. Furthermore, the Debye shielding over distances $\lambda_D \ll l_c$ attenuates the local fluctuations of the electric field within the small l_c^3 volume. PLE plasmas can be also described using hydrodynamic transport equations for distances L_c much longer than the scale length l_c introduced by the dominant collisional process.

Box 3.1 shows that a number of plasmas in Nature and in the laboratory where local fluctuations decay over small distances l_c compared with the characteristic transport length scale L_s satisfy these conditions. The opposite limit where $\lambda_c \gtrsim L_s$ requires a kinetic approach using the Boltzmann or Vlasov equations.

The fluid approximation assumes smooth changes of the non-equilibrium velocity distribution function $f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)$ along the length $L_s \gg l_c$ and time $T_s \gg l_c/|\boldsymbol{u}_\alpha|$ scales involved in the macroscopic plasma transport. In these conditions the *hydrodynamic plasma transport equations* governing the temporal and spatial evolution of $n_\alpha(\boldsymbol{r}, t)$, $\boldsymbol{u}_\alpha(\boldsymbol{r}, t)$ and $T_\alpha(\boldsymbol{r}, t)$, can be directly derived from the Boltzmann equation (2.10)

[2] The equivalent time-independent averages were introduced in section 3.3 of volume 1.

[3] Strictly speaking, the Debye length expression (4.13) in volume 1 is derived for Maxwellian plasmas. This is not the general case, so in its place we can consider λ_D as an *effective length scale* for electric field shielding along which plasma quasineutrality is preserved.

Box 3.1. The fluid approximation

A high pressure argon arc has a typical kinetic temperature $T \sim 0.5$ eV and plasma density $n_e \sim 10^{15}$ cm^{-3} for the gas pressure of $p_a \sim 50$ mbar in figures 1.2 and 4.1 of volume 1. This partially ionized ($\alpha_g \sim 0.08\%$) LTE plasma has $\lambda_D \sim 1.7 \times 10^{-5}$ cm and in figure 6.3(a) of volume 1 the cross section is of the order of $\sigma_{el} \sim 10^{-15}$ cm^2. These values give the mean free path $\lambda_c \sim 8.3 \times 10^{-4}$ cm for the short-range elastic collisions.

Therefore, for the length scale $l_c \sim 10^{-2}$ cm $\gtrsim \lambda_D \sim 10^{-3}$ cm the number of electrons inside l_c^3 is,

$$N_e = l_c^3 \times n_e = (10^{-2})^3 \times 10^{15} = 10^9 \text{ charged particles.}$$

Therefore, the fluid approximation $L_s \gg l_c \gtrsim \lambda_D$ remains valid when the macroscopic dimensions of the physical system are $L_s \gtrapprox 0.1$ cm, which is small for the length scales involved in mass transport in arc plasmas.

The density of charged particles in figure 1.7 of volume 1 for the ionospheric F-peak located at 200 km of altitude is $n \sim 2.5 \times 10^5$ particles by cubic centimeter with $T_e \sim 0.05$ eV which gives $\lambda_D \sim 0.3$ cm (see table 4.1, volume 1). The average of the ion mass at this height is approximated by 24 amu[a] [1] and in this fully ionized plasma, the long-range Coulomb collision frequency of an electron by ions can be estimated using equation (5.22) of volume 1,

$$\nu_{ei} = \frac{e^4\, n_e \ln \Lambda_{cl}}{6\sqrt{2}\,\epsilon_o^2\,\pi^{3/2}\, m_e^{1/2}\,(k_B T)^{3/2}} \quad \text{and} \quad \nu_{ie} \sim \frac{m_e}{m_i}\,\nu_{ei} \ll \nu_{ei}$$

for the deflection of ions by electrons. Using for the Coulomb logarithm $\ln \lambda_c = 15$ we obtain $\nu_{ie} \sim 0.46$ s^{-1} and $\nu_{ei} \sim 10^{-3}$ s^{-1}, hence $\lambda_{ie} = c_{is}/\nu_{ie} \sim 4$ km, where c_{is} is the ion sound speed.

Hence, $l_c \sim 4 \times 10^5$ cm $\gg \lambda_D \sim 0.3$ cm and the number of electrons inside l_c^3 is now,

$$N_e = l_c^3 \times n_e = (10^5)^3 \times 2.5 \times 10^5 = 2.5 \times 10^{20} \text{ charged particles.}$$

In this case, the macroscopic length scale $L_s \gg l_c \gtrsim \lambda_D$ is much longer than the collision mean free path for ions, in the order of tens of kilometers. However, this value is small compared with the the characteristic lengths involved in the ionospheric mass transport.

without using an explicit expression for the distribution function. However, the specific details of particle collisions considered in the operator $C(f_\alpha)$ are required to derive the macroscopic transport coefficients, such as thermal conductivity, viscosity, etc.

[a] 1 amu = 1.66×10^{-27} kg.

3.1 The moments of Boltzmann equation

The *macroscopic transport equations* that govern the temporal and spatial variations of density, fluid velocity and kinetic temperature can be derived by the mathematical procedure of *taking moments* of the Boltzmann equation (2.10). The *moment* $\boldsymbol{M}^{(k)}(\boldsymbol{r}, t)$ of order k of the distribution function $f(\boldsymbol{v}, \boldsymbol{r}, t)$ can be formally written as,

$$\boldsymbol{M}^{(k)}(\boldsymbol{r}, t) = \int_{-\infty}^{+\infty} \underbrace{(\boldsymbol{v} \otimes \boldsymbol{v} \otimes \ldots \otimes \boldsymbol{v})}_{\text{k times}} f(\boldsymbol{v}, \boldsymbol{r}, t)\, d\boldsymbol{v} \tag{3.1}$$

where $\boldsymbol{a} \otimes \boldsymbol{b}$ denotes the tensor product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. The infinite set of moments $\boldsymbol{M}^{(k)}(\boldsymbol{r}, t)$ for $k = 0, 1, 2, \ldots$ characterizes the function $f(\boldsymbol{v}, \boldsymbol{r}, t)$ when this distribution is smooth enough.

These *moments* are the mathematical generalization of the averages in the phase space introduced in chapter 3 of volume 1 and chapter 2 of this volume. For the particle species α these averages in the $(\boldsymbol{r}, \boldsymbol{v})$ space can be generalized for the time dependent $H_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)$ function as,

$$\langle H_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)\rangle = \frac{1}{n_\alpha(\boldsymbol{r}, t)} \int_{-\infty}^{+\infty} H_\alpha(\boldsymbol{v}, \boldsymbol{r}, t) f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)\, d\boldsymbol{v} \tag{3.2}$$

where again the index ($\alpha = a, e, i$) indicates neutral atoms, electrons or ions.

The connection of these abstract mathematical expressions with magnitudes of physical interest can be grasped with the simplest choice $H_\alpha(\boldsymbol{v}, \boldsymbol{r}, t) = 1$ corresponding to $k = 0$ in equation (3.2). We obtain,

$$1 = \frac{1}{n_\alpha(\boldsymbol{r}, t)} \int_{-\infty}^{+\infty} f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)\, d\boldsymbol{v} \quad \text{or,} \quad n_\alpha(\boldsymbol{r}, t) = \int_{-\infty}^{+\infty} f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)\, d\boldsymbol{v} \tag{3.3}$$

which is related with the number density $n_\alpha(\boldsymbol{r}, t)$ introduced in equation (2.4). Equivalently,

$$1 = \int_{-\infty}^{+\infty} P_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)\, d\boldsymbol{v}$$

is the normalization of the probability distribution function $P_\alpha = f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)/n_\alpha(\boldsymbol{r}, t)$ introduced in equation (2.2). Setting $k = 1$ in equation (3.1) or $H_\alpha(\boldsymbol{v}, \boldsymbol{r}, t) = \boldsymbol{v}$ in equation (3.2) we have,

$$\langle \boldsymbol{v}\rangle = \frac{1}{n_\alpha(\boldsymbol{r}, t)} \int_{-\infty}^{+\infty} \boldsymbol{v}\, f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)\, d\boldsymbol{v} \tag{3.4}$$

Then $\langle \boldsymbol{v}\rangle = \boldsymbol{u}_\alpha(\boldsymbol{r}, t)$ is the average macroscopic fluid velocity introduced in equation (2.5). With $H_\alpha(\boldsymbol{v}, \boldsymbol{r}, t) = m_\alpha\,(\boldsymbol{v} \otimes \boldsymbol{v})$ or $k = 2$ in equation (3.1) we have,

$$m_\alpha \langle \boldsymbol{v} \otimes \boldsymbol{v}\rangle = \frac{1}{n_\alpha(\boldsymbol{r}, t)} \int_{-\infty}^{+\infty} m_\alpha\,(\boldsymbol{v} \otimes \boldsymbol{v}) f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)\, d\boldsymbol{v} \tag{3.5}$$

and the components of the *momentum flow tensor* $\boldsymbol{S}_\alpha = m_\alpha\, n_\alpha(\boldsymbol{r},\, t)\, \langle \boldsymbol{v} \otimes \boldsymbol{v} \rangle$ are,

$$\begin{pmatrix} v_x^2 & v_x v_y & v_x v_z \\ v_y v_x & v_y^2 & v_y v_z \\ v_z v_x & v_z v_y & v_z^2 \end{pmatrix} \tag{3.6}$$

where it is clear that $S_{i\,j} = S_{j\,i}$ so the matrix is symmetric.

The scalar function $H_\alpha(\boldsymbol{v},\, \boldsymbol{r},\, t) = m_\alpha\, (\boldsymbol{v} \cdot \boldsymbol{v})/2 = m_\alpha v^2/2$ gives the average kinetic energy per particle,

$$\left\langle \frac{m_\alpha\, v^2}{2} \right\rangle = \frac{1}{n_\alpha(\boldsymbol{r},\, t)} \int_{-\infty}^{\infty} \left(\frac{m_\alpha\, v^2}{2} \right) f_\alpha(\boldsymbol{v},\, \boldsymbol{r},\, t)\, d\boldsymbol{v} \tag{3.7}$$

whereas $H_\alpha(\boldsymbol{v},\, \boldsymbol{r},\, t) = (m_\alpha v^2/2)\, \boldsymbol{v}$ leads to,

$$\left\langle \left(\frac{m_\alpha\, v^2}{2} \right) \boldsymbol{v} \right\rangle = \frac{1}{n_\alpha(\boldsymbol{r},\, t)} \int_{-\infty}^{+\infty} \left(\frac{m_\alpha\, v^2}{2} \right) \boldsymbol{v}\, f_\alpha(\boldsymbol{v},\, \boldsymbol{r},\, t)\, d\boldsymbol{v} \tag{3.8}$$

which represents the average *flux of kinetic energy* transported per particle.

The physical description of macroscopic fluids is therefore related to the first moments of equation (3.1), that are connected to the averages in phase space over the velocity distribution function. At this point we face for the first time the problem of the *closure* of transport equations as the rigorous mathematical calculation requires the determination of an *infinite number of moments* given by equation (3.1). The time evolution of n_α depends on the velocity $\boldsymbol{u}_\alpha$, which in turn is a function of $\boldsymbol{S}_\alpha$ and so on; each one depends on the following higher order moment.

However, the hydrodynamic transport equations make use only of few averages (3.3)–(3.16), which are related to the magnitudes of physical significance. As we shall see, the derivation of transport equations requires a *truncation scheme* or further physical assumptions to cope with the reduced number of moments (3.1) of physical interest.

Before deriving the hydrodynamic equations of a plasma we will introduce few additional concepts. First, we consider the infinitesimal volume element P of figure 3.1 centered at point $\boldsymbol{r}$ of small size l_c but big enough to contain a large number $n_\alpha\, l_c^3 \gg 1$ of particles inside. Its center of mass moves with respect to the laboratory frame with average velocity $\boldsymbol{u}_\alpha(\boldsymbol{r},\, t)$ (2.5) or (3.4) along the dotted path line.

The velocity of a specific particle $\boldsymbol{v} = \boldsymbol{c} + \boldsymbol{u}_\alpha$ can be expressed as the sum of $\boldsymbol{u}_\alpha$ and a *randomly fluctuating component* $\boldsymbol{c}$ called *peculiar velocity* relative to a frame that moves with P with the $\boldsymbol{u}_\alpha$ velocity. Since $\boldsymbol{c}$ characterized the random motion of particles with respect to P its average[4] (3.2),

[4] We have $d\boldsymbol{v} = d\boldsymbol{c}$ and we also write $d\boldsymbol{v} = d^3v = d\boldsymbol{c} = d^3c$. Equivalently $\langle \boldsymbol{v} \rangle = \langle \boldsymbol{c} \rangle + \langle \boldsymbol{v} \rangle$ gives $\boldsymbol{u}_\alpha = \langle \boldsymbol{c} \rangle + \boldsymbol{u}_\alpha$ and then $\langle \boldsymbol{c} \rangle = 0$.

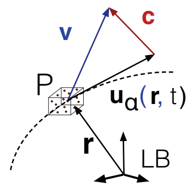

Figure 3.1. The small volume P moves with respect to the laboratory frame.

$$\langle \boldsymbol{c} \rangle = \frac{1}{n_\alpha(\boldsymbol{r},\, t)} \int_{-\infty}^{+\infty} \boldsymbol{c}\, f_\alpha(\boldsymbol{v},\, \boldsymbol{r},\, t)\, d\boldsymbol{v} = 0$$

is null. In contrast, $\langle c^2 \rangle$ is related to the average kinetic energy with respect to the center of mass of P and,

$$\left\langle \frac{m_\alpha}{2} \boldsymbol{c}^2 \right\rangle = \frac{1}{n_\alpha(\boldsymbol{r},\, t)} \int_{-\infty}^{+\infty} \frac{m_\alpha}{2} c^2 f_\alpha(\boldsymbol{v},\, \boldsymbol{r},\, t)\, d\boldsymbol{v} = \frac{m_\alpha}{2} \langle c^2 \rangle > 0 \tag{3.9}$$

is positive since particles are always in thermal motion. For the average velocity (3.4) we can write,

$$\langle v^2 \rangle = \langle (\boldsymbol{u}_\alpha + \boldsymbol{c})^2 \rangle = \langle \boldsymbol{u}_\alpha^2 + 2\,(\boldsymbol{u}_\alpha \cdot \boldsymbol{c}) + \boldsymbol{c}^2 \rangle = u_\alpha^2 + \langle c^2 \rangle$$

Using equation (3.7) the average kinetic energy is,

$$E_\alpha(\boldsymbol{r},\, t) = \int_{-\infty}^{\infty} \left(\frac{m_\alpha\, v^2}{2} \right) f_\alpha(\boldsymbol{v},\, \boldsymbol{r},\, t)\, d\boldsymbol{v} = n_\alpha(\boldsymbol{r},\, t) \left\langle \frac{m_\alpha\, v^2}{2} \right\rangle$$

and we have,

$$E_\alpha = n_\alpha \left\langle \frac{m_\alpha}{2}\, v^2 \right\rangle = n_\alpha \frac{m_\alpha}{2} \left(u_\alpha^2 + \langle c^2 \rangle\right) = m_\alpha\, n_\alpha \frac{u_\alpha^2}{2} + e_{i\alpha} \tag{3.10}$$

The first term represents the average kinetic energy of particle by unit of volume with respect to the laboratory frame. The second adding $e_{i\alpha} = m_\alpha\, n_\alpha \langle c^2 \rangle / 2$ in equation (3.10) involves the random *peculiar velocity* and is the kinetic energy per volume unit in a frame moving with the small volume P and is a function of the kinetic temperature. Equation (3.10) states that increments in the energy of the small volume P, can raise its internal energy $e_{i\alpha}$ (temperature T_α), its velocity $\boldsymbol{u}_\alpha$ with respect to the laboratory frame or both.

When the particles in P can be considered in a *local equilibrium* $e_{i\alpha} \sim T_\alpha$ and for a local Maxwellian distribution (4.3) of volume 1, we have the internal energy[5],

$$\left\langle \frac{m_\alpha\, c^2}{2} \right\rangle = \frac{3}{2} k_B\, T_\alpha(\boldsymbol{r},\, t) \quad \text{and,} \quad e_{i\alpha} = \frac{3}{2} n_\alpha(\boldsymbol{r},\, t)\, k_B\, T_\alpha(\boldsymbol{r},\, t) \tag{3.11}$$

[5] We make use of the internal energy of a monoatomic ideal, for molecules the factor 3/2 is replaced by $1/(\gamma - 1)$ where γ is the adiabatic index.

Equivalently, this medium is an LTE plasma where the equation,

$$p_\alpha(\boldsymbol{r},\, t) = n_\alpha(\boldsymbol{r},\, t)\, k_B T_\alpha(\boldsymbol{r},\, t)$$

is valid as far as short and long-range collisions relax fluctuations back to the stationary equilibrium state much faster than the $l_s/|\boldsymbol{u}_\alpha|$ slow time scale. This principle also applies to PLE plasmas under appropriate length and time scales.

Introducing $\boldsymbol{v} = \boldsymbol{u}_\alpha + \boldsymbol{c}$ in the expression for $\boldsymbol{S}_\alpha$, the averages in the matrix (3.5) are of the form,

$$\langle (u_{\alpha i} + c_i)(u_{\alpha j} + c_j) \rangle = \langle u_{\alpha i}\, u_{\alpha j} \rangle + \langle c_i\, c_j \rangle + u_{\alpha i}\, \langle c_j \rangle + u_{\alpha j}\, \langle c_i \rangle$$

Since $\langle c_i \rangle = 0$ are null for the components of the random velocity we have,

$$\boldsymbol{S}_\alpha = m_\alpha\, n_\alpha\, (\boldsymbol{u}_\alpha \otimes \boldsymbol{u}_\alpha) + m_\alpha\, n_\alpha \langle \boldsymbol{c} \otimes \boldsymbol{c} \rangle$$

and $\boldsymbol{S}_\alpha$ is called *momentum flow tensor* for the α-particles. The second is the *kinetic pressure tensor* $\boldsymbol{P}_\alpha$ term,

$$\boldsymbol{P}_\alpha = m_\alpha\, n_\alpha \langle \boldsymbol{c} \otimes \boldsymbol{c} \rangle = \boldsymbol{D}_\alpha + \boldsymbol{\Pi}_\alpha$$

and its components are similar to the matrix (3.6) replacing v_{ij} by c_{ij}. It can be cast as the sum of the $\boldsymbol{D}_\alpha$ and $\boldsymbol{\Pi}_\alpha$ tensors where,

$$\boldsymbol{D}_\alpha = m_\alpha\, n_\alpha \begin{pmatrix} \langle c_y^2 \rangle & 0 & 0 \\ 0 & \langle c_y^2 \rangle & 0 \\ 0 & 0 & \langle c_y^2 \rangle \end{pmatrix} \tag{3.12}$$

Under LTE conditions the distribution of random peculiar velocities $\boldsymbol{c}$ is *isotropic* so,

$$\langle c_x^2 \rangle = \langle c_y^2 \rangle = \langle c_z^2 \rangle = \frac{\langle c^2 \rangle}{3}$$

and the tensor $\boldsymbol{D}_\alpha$ becomes proportional to a diagonal matrix. Using the ideal gas approximation (3.11) we have $\langle c^2 \rangle = 3\, k_B\, T_\alpha / m_\alpha$ and for one component,

$$m_\alpha\, n_\alpha\, \langle c_x^2 \rangle = m_\alpha\, n_\alpha\, \frac{\langle c^2 \rangle}{3} = n_\alpha k_B\, T_\alpha$$

The pressure $p_\alpha(\boldsymbol{r},\, t)$ in equation (3.12),

$$p_\alpha(\boldsymbol{r},\, t) = \frac{m_\alpha}{3} \int_{-\infty}^{+\infty} c^2 f_\alpha(\boldsymbol{v},\, \boldsymbol{r},\, t)\, d\boldsymbol{v} = \frac{m_\alpha n_\alpha}{3} \langle c^2 \rangle \tag{3.13}$$

represents the scalar *partial pressure* due to the particles of type α. The second tensor $\boldsymbol{\Pi}_\alpha$ is related with viscous stresses as in conventional fluid theory and its components are,

$$\mathbf{\Pi}_\alpha = m_\alpha\, n_\alpha \begin{pmatrix} 0 & \langle c_x\, c_y\rangle & \langle c_x\, c_z\rangle \\ \langle c_y\, c_x\rangle & 0 & \langle c_y\, c_z\rangle \\ \langle c_z\, c_x\rangle & \langle c_z\, c_y\rangle & 0 \end{pmatrix} \tag{3.14}$$

where $\langle c_i\, c_j\rangle = \langle c_j\, c_i\rangle$. Finally, the *momentum flow tensor* reads,

$$\boldsymbol{S}_\alpha = m_\alpha\, n_\alpha\, \langle \boldsymbol{v} \otimes \boldsymbol{v}\rangle = m_\alpha\, n_\alpha\, (\boldsymbol{u}_\alpha \otimes \boldsymbol{u}_\alpha) + \left(p_\alpha\, \boldsymbol{I}\right) + \mathbf{\Pi}_\alpha \tag{3.15}$$

where $\boldsymbol{I}$ is the unity matrix and the ideal gas law relation was also employed, equivalent to a local Maxwellian velocity distribution function for the LTE equilibrium of α-particles.

The non-diagonal terms can be neglected as the averages $\langle c_i\, c_j\rangle$ in tensor $\mathbf{\Pi}_\alpha$ are small under LTE conditions and $\boldsymbol{P}_\alpha \simeq (p_\alpha(\boldsymbol{r},\, t)\, \boldsymbol{I})$ for isotropic velocity distributions of peculiar (random) velocities.

The *energy density flux* vector $\boldsymbol{K}_\alpha(\boldsymbol{r},\, t)$ is introduced from equation (3.8) as,

$$\boldsymbol{K}_\alpha(\boldsymbol{r},\, t) = \int_{-\infty}^{+\infty} \left(\frac{m_\alpha\, v^2}{2}\right) \boldsymbol{v}\, f_\alpha(\boldsymbol{v},\, \boldsymbol{r},\, t)\, d\boldsymbol{v} = \frac{m_\alpha n_\alpha}{2}\, \langle v^2\, \boldsymbol{v}\rangle \tag{3.16}$$

and using $\boldsymbol{v} = \boldsymbol{c} + \boldsymbol{u}_\alpha$ in the average, the non-null terms are,

$$\langle v^2\, \boldsymbol{v}\rangle = (u_\alpha^2 + \langle c^2\rangle)\, \boldsymbol{u}_\alpha + \langle c^2\, \boldsymbol{c}\rangle + 2\, \langle\, (\boldsymbol{c} \cdot \boldsymbol{u}_\alpha)\, \boldsymbol{c}\rangle$$

The first addition at the right side is proportional to the transport $E_\alpha\, \boldsymbol{u}_\alpha$ of energy introduced in equation (3.10). The second term is the *heat flux vector* $\boldsymbol{q}_\alpha$ for the α species which is defined as,

$$\boldsymbol{q}_\alpha = \int_{-\infty}^{+\infty} \left(\frac{m_\alpha c^2}{2}\right) \boldsymbol{c}\, f_\alpha(\boldsymbol{v},\, \boldsymbol{r},\, t)\, d\boldsymbol{v} = \frac{m_\alpha\, n_\alpha}{2}\, \langle c^2\, \boldsymbol{c}\rangle \tag{3.17}$$

The vector $\boldsymbol{A} = \langle (\boldsymbol{c} \cdot \boldsymbol{u}_\alpha)\, \boldsymbol{c}\rangle$ in the last term is,

$$\boldsymbol{A} = \left\langle \left(\sum_j c_j\, u_{\alpha j} \right) \boldsymbol{c} \right\rangle$$

and for its k component of $\boldsymbol{c}$ we have,

$$A_k = \left\langle \left(\sum_j c_j\, u_{\alpha j} \right) c_k \right\rangle = \sum_j \langle c_j\, c_k\rangle\, u_{\alpha j}$$

this is equivalent to the contraction operation of the symmetric second order *momentum flux tensor* $m_\alpha\, n_\alpha \langle \boldsymbol{c} \otimes \boldsymbol{c}\rangle$ sum of equations (3.12) and (3.12) with the velocity $\boldsymbol{u}_\alpha$ we denote[6] as,

[6] Here the operation $\boldsymbol{A}$: $\boldsymbol{B}$ denotes the contraction of second order tensor $\boldsymbol{A}$ with vector $\boldsymbol{B}$.

$$\boldsymbol{A} = \langle(\boldsymbol{c} \cdot \boldsymbol{u}_\alpha)\, \boldsymbol{c}\rangle = (P_\alpha \boldsymbol{I} + \boldsymbol{\Pi}_\alpha) : \boldsymbol{u}_\alpha$$

and $p_\alpha\ (\mathbf{1}\!: \boldsymbol{u}_\alpha) = (p_\alpha\ \boldsymbol{u}_\alpha)$. Finally, the *energy density flux vector* becomes,

$$\boldsymbol{K}_\alpha(\boldsymbol{r},\ t) = E_\alpha\, \boldsymbol{u}_\alpha + \boldsymbol{q}_\alpha + (p_\alpha\ \boldsymbol{u}_\alpha) + (\boldsymbol{\Pi}_\alpha\!: \boldsymbol{u}_\alpha) \tag{3.18}$$

The following derivation of the macroscopic transport equations needs equations (3.15), (3.17) and (3.18).

3.1.1 The equation of continuity

The equation for the conservation of the number (or mass density) of particles for the α species is obtained integrating equation (2.10) over the velocities $\boldsymbol{v}$ in the phase space. We have,

$$\int_{-\infty}^{+\infty} \frac{\partial f_\alpha}{\partial t}\, d\boldsymbol{v} + \int_{-\infty}^{+\infty} \nabla_{\mathbf{r}} \cdot (\boldsymbol{v}\, f_\alpha)\, d\boldsymbol{v} + \int_{-\infty}^{+\infty} \nabla_{\mathbf{v}} \cdot \left(\frac{\boldsymbol{F}_{e\alpha}}{m_\alpha} f_\alpha\right) d\boldsymbol{v} = \int_{-\infty}^{+\infty} C_s(f_\alpha)\, d\boldsymbol{v}$$

where $\boldsymbol{F}_{e\alpha} = q_\alpha\ [\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B})]$ is the Lorentz force acting on the α-particle and $\boldsymbol{E}(\boldsymbol{r},\ t)$ and $\boldsymbol{B}(\boldsymbol{r},\ t)$ the electromagnetic fields. Integrals in two first terms on the right can be directly permuted with the operators $\partial/\partial t$ and $\nabla_{\mathbf{r}}$ using the divergence theorem for the third adding we have,

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} f_\alpha\, d\boldsymbol{v} + \nabla_{\mathbf{r}} \cdot \int_{-\infty}^{+\infty} (\boldsymbol{v} f_\alpha)\, d\boldsymbol{v} + \underbrace{\int_{s(v)} \left(\frac{\boldsymbol{F}_{e\alpha}}{m_\alpha} f_\alpha\right) \cdot d\boldsymbol{S}}_{\to 0} = \int_{-\infty}^{+\infty} C_s(f_\alpha)\, d\boldsymbol{v}$$

The third integral is null since the velocity distribution function decreases as the velocity $\boldsymbol{v}$ grows increasing the surface $S(v)$ in the velocity space and equations (2.4) and (2.5) give,

$$n_\alpha(\boldsymbol{r},\ t) = \int_{-\infty}^{+\infty} f_\alpha(\boldsymbol{v},\ \boldsymbol{r},\ t)\, d\boldsymbol{v} \quad \text{and} \quad n_\alpha\, \boldsymbol{u}_\alpha = \int_{-\infty}^{+\infty} \boldsymbol{v}\, f_\alpha(\boldsymbol{v},\ \boldsymbol{r},\ t)\, d\boldsymbol{v}$$

Finally we obtain for the macroscopic density of α species,

$$\frac{\partial n_\alpha}{\partial t} + \nabla_{\mathbf{r}} \cdot (n_\alpha \boldsymbol{u}_\alpha) = \int_{-\infty}^{+\infty} C_\alpha(f_\alpha)\, d\boldsymbol{v}$$

where the right term is the moment of the collisional operator. In section 2.3 we introduced equation (2.16) as the number of α-particles added or removed in time δt from the phase space volume $d^3v\ d^3r$ located at point $(\boldsymbol{r},\ t)$ due to collisions with the β species. We can write,

$$C_\alpha = \frac{\delta N_\alpha^{\text{in}} - \delta N_\alpha^{\text{out}}}{d^3v\ d^3r\ \delta t}$$

for the creation and recombination of α-particles. Then,

$$\int_{-\infty}^{+\infty} C_\alpha(f_\alpha)\, d\boldsymbol{v} = \frac{\delta N_\alpha^{\text{in}} - \delta N_\alpha^{\text{out}}}{d^3r\ \delta t} = \frac{\delta n_\alpha^{\text{in}}}{\delta t} - \frac{\delta n_\alpha^{\text{out}}}{\delta t} = S_\alpha - L_\alpha$$

where S_α and L_α represent the gain and loss rates per unit volume of geometry space. In the case of ions and electrons, these are the ionization $S_i = \langle k_I \rangle \, n_a \, n_e$ and recombination $L_i = \beta_R \, n_e \, n_i$ collision frequencies, equations (6.4) and (6.5) in volume 1. In the plasma equilibrium, the α-particles are lost and created at equal rates and $S_\alpha = L_\alpha$ expresses the balance of the corresponding forward and backward elementary processes.

Finally, the fluid particle continuity equation for α-particles reads,

$$\frac{\partial n_\alpha}{\partial t} + \nabla_{\mathbf{r}} \cdot (n_\alpha \, \boldsymbol{u}_\alpha) = S_\alpha - L_\alpha \tag{3.19}$$

The continuity equation expresses the conditions for the conservation of (α= a, e, i) particles in each fluid element, similarly to conventional hydrodynamics. In this equation the different plasma species are coupled through the particle source/sink terms on the right hand. In this derivation no explicit expression for the velocity distribution function $f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)$ has been used. The statistical averages (2.4) and (2.5) only require the assumption of a bounded smooth $f_\alpha(\boldsymbol{v}_\alpha, \boldsymbol{r}, t)$ velocity distribution function.

In contrast, the calculation of S_α and L_α in equations (6.4) and (6.5) of volume 1 need an *explicit expression* for $f_\alpha(\boldsymbol{v}_\alpha, \boldsymbol{r}, t)$ and physical models for elementary processes such as the ionizing, recombination collisions, etc, at the atomic and molecular level. Therefore, as is indicated in the scheme of figure 1.1, the *mathematical closure* of plasma fluid equations will require *additional information* and/or further approximations.

3.1.2 The momentum transport equation

The next moment of the Boltzmann equation with physical meaning is obtained by multiplying the equation (2.10) by $m_\alpha \boldsymbol{v}$ and integrating over the velocities $\boldsymbol{v}$ as,

$$\begin{aligned} \int_{-\infty}^{+\infty} (m_\alpha \, \boldsymbol{v}) \frac{\partial f_\alpha}{\partial t} \, d\boldsymbol{v} &+ \int_{-\infty}^{+\infty} (m_\alpha \, \boldsymbol{v}) \, \nabla_{\mathbf{r}} \cdot (\boldsymbol{v} \, f_\alpha) \, d\boldsymbol{v} \\ &+ \int_{-\infty}^{+\infty} (m_\alpha \, \boldsymbol{v}) \, \nabla_{\mathbf{v}} \cdot (\boldsymbol{F}_{e\alpha} \, f_\alpha) \, d\boldsymbol{v} = \\ &\qquad \int_{-\infty}^{+\infty} (m_\alpha \, \boldsymbol{v}) \, C_\alpha(f_\alpha) \, d\boldsymbol{v} \end{aligned} \tag{3.20}$$

where $\boldsymbol{F}_{e\alpha} = q_\alpha \, [\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B})]$ is the Lorentz force acting on the α-particle. The right term involves the collision operator $C_s(f_\alpha)$ and can be simply expressed as the vector,

$$\boldsymbol{R}_\alpha = \int_{-\infty}^{+\infty} (m_\alpha \, \boldsymbol{v}) \, C_\alpha(f_\alpha) \, d\boldsymbol{v} \tag{3.21}$$

that will be discussed later in section 3.1.3.

The first term in equation (3.20) is proportional to the partial derivative of the particle number flux (2.6) and,

$$\int_{-\infty}^{+\infty} (m_\alpha v) \frac{\partial f_\alpha}{\partial t} dv = m_\alpha \frac{\partial}{\partial t}(n_\alpha \langle v \rangle) = m_\alpha \frac{\partial}{\partial t}(n_\alpha \boldsymbol{u}_\alpha)$$

The second addition on the left hand of equation (3.20) is transformed using the following vector identity,

$$\nabla_{\mathbf{r}} \cdot [\boldsymbol{v} \otimes (\boldsymbol{v} f_\alpha)] = [\boldsymbol{v} \cdot \nabla_{\mathbf{r}}](\boldsymbol{v} f_\alpha) + \underbrace{[(f_\alpha \boldsymbol{v}) \cdot \nabla_{\mathbf{r}}]\, \boldsymbol{v}}_{0} = (\boldsymbol{v} \cdot \nabla_{\mathbf{r}})(\boldsymbol{v} f_\alpha)$$

and we have,

$$\int_{-\infty}^{+\infty} (m_\alpha v) \nabla_{\mathbf{r}} \cdot (v f_\alpha)\, dv = \nabla_{\mathbf{r}} \cdot \left[\int_{-\infty}^{+\infty} m_\alpha (v \otimes v) f_\alpha \, dv\right]$$
$$= \nabla_{\mathbf{r}} \cdot (m_\alpha n_\alpha \langle v \otimes v \rangle)$$

where we can substitute the *momentum flux tensor* $\boldsymbol{S}_\alpha(\boldsymbol{r}, t) = m_\alpha n_\alpha \langle v \otimes v \rangle$ of equation (3.15).

The integral of the third term on the left side of equation (3.20) can be also transformed using a vector identity and $(\boldsymbol{F}_{e\alpha} f_\alpha \cdot \nabla_{\mathbf{v}})\, v = f_\alpha \boldsymbol{F}_{e\alpha}$ as,

$$\nabla_{\mathbf{v}} \cdot [(\boldsymbol{F}_{e\alpha} f_\alpha) \otimes v] = (v \cdot \nabla_{\mathbf{v}})(\boldsymbol{F}_{e\alpha} f_\alpha) + (\boldsymbol{F}_{e\alpha} f_\alpha \cdot \nabla_{\mathbf{v}})\, v = v\, \nabla_{\mathbf{v}} \cdot (\boldsymbol{F}_{e\alpha} f_\alpha) + f_\alpha \boldsymbol{F}_{e\alpha}$$

and the third integral in equation (3.20) now reads,

$$\int_{-\infty}^{+\infty} \boldsymbol{v} \nabla_{\mathbf{v}} \cdot (\boldsymbol{F}_{e\alpha} f_\alpha)\, d\boldsymbol{v} = \underbrace{\int_{-\infty}^{+\infty} \nabla_{\mathbf{v}} \cdot [f_\alpha (\boldsymbol{v} \otimes \boldsymbol{F}_{e\alpha})]\, d\boldsymbol{v}}_{0} - \int_{-\infty}^{+\infty} f_\alpha \boldsymbol{F}_{e\alpha}\, d\boldsymbol{v}$$

Using the divergence theorem, the first addition on the right hand is null assuming the distribution function f_α decreases monotonically as the velocity v grows. Only the second term remains and for the Lorentz force action on the α-particles we have,

$$\int_{-\infty}^{+\infty} f_\alpha \boldsymbol{F}_{e\alpha}\, dv = q_\alpha \boldsymbol{E} \left(\int_{-\infty}^{+\infty} f_\alpha \, dv\right) + q_\alpha \left(\left[\int_{-\infty}^{+\infty} v f_\alpha \, dv\right] \times \boldsymbol{B}\right)$$

Using again equations (2.4) and (2.5) we can write,

$$n_\alpha \langle \boldsymbol{F}_{e\alpha} \rangle = q_\alpha n_\alpha [\boldsymbol{E} + (\boldsymbol{u}_\alpha \times \boldsymbol{B})] = \rho_{e\alpha} \boldsymbol{E} + \boldsymbol{J}_{e\alpha} \times \boldsymbol{B}$$

where $\rho_{e\alpha} = q_\alpha n_\alpha$ is the electric charge density and $\boldsymbol{J}_{e\alpha} = q_\alpha n_\alpha \boldsymbol{u}_\alpha$ electric current density of α-particles that also appear in the Maxwell equations.

With these changes the equation for momentum transport derived from the moment (3.20) now reads,

$$m_\alpha \frac{\partial}{\partial t}(n_\alpha \boldsymbol{u}_\alpha) + \nabla_{\mathbf{r}} \cdot \boldsymbol{S}_\alpha = \rho_{e\alpha} \boldsymbol{E} + \boldsymbol{J}_{e\alpha} \times \boldsymbol{B} + \boldsymbol{R}_\alpha \tag{3.22}$$

We can further transform this equation using divergence of the momentum flux tensor $\boldsymbol{S}_\alpha$ (3.15) as,

$$\nabla_{\mathbf{r}} \cdot \boldsymbol{S}_\alpha = \nabla_{\mathbf{r}} \cdot [n_\alpha (\boldsymbol{u}_\alpha \otimes \boldsymbol{u}_\alpha)] + \nabla_{\mathbf{r}} \cdot (p_\alpha \boldsymbol{I}) + \nabla_{\mathbf{r}} \cdot \boldsymbol{\Pi}_\alpha$$

and,

$$\nabla_{\mathrm{r}} \cdot \boldsymbol{S}_\alpha = n_\alpha(\boldsymbol{u}_\alpha \cdot \nabla_{\mathrm{r}})\, \boldsymbol{u}_\alpha + \boldsymbol{u}_\alpha\, [\nabla_{\mathrm{r}} \cdot (n_\alpha \boldsymbol{u}_\alpha)] + \nabla_{\mathrm{r}}\, p_\alpha + \nabla_{\mathrm{r}} \cdot \boldsymbol{\Pi}_\alpha$$

Substituting in equation (3.22) we have,

$$\begin{aligned} &m_\alpha n_\alpha \frac{\partial \boldsymbol{u}_\alpha}{\partial t} + m_\alpha \boldsymbol{u}_\alpha \left[\frac{\partial n_\alpha}{\partial t} + \nabla_{\mathrm{r}} \cdot (n_\alpha \boldsymbol{u}_\alpha)\right] \\ &\quad + m_\alpha n_\alpha (\boldsymbol{u}_\alpha \nabla_{\mathrm{r}} \cdot\,)\, \boldsymbol{u}_\alpha + \nabla_{\mathrm{r}}\, p_\alpha + \nabla_{\mathrm{r}} \cdot \boldsymbol{\Pi}_\alpha = (\rho_{e\alpha}\, \boldsymbol{E} + \boldsymbol{J}_{e\alpha} \times \boldsymbol{B} + \boldsymbol{R}_\alpha) \end{aligned} \tag{3.23}$$

The square brackets contain the continuity equation (3.19) and,

$$m_\alpha \boldsymbol{u}_\alpha \left[\frac{\partial n_\alpha}{\partial t} + \nabla_{\mathrm{r}} \cdot (n_\alpha \boldsymbol{u}_\alpha)\right] = m_\alpha\, \boldsymbol{u}_\alpha\, (S_\alpha - L_\alpha)$$

this contribution accounts for the momentum change associated with the creation/destruction of α-particles. For example, a pair electron/ion is produced in the ionization event of a neutral atom, consequently the ion ($\alpha = i$) and electron ($\alpha = e$) *fluids* experience a change in momentum.

Finally, the momentum transport equation[7] takes a simpler equivalent form and for the ($\alpha = e, i, a$) plasma particles is,

$$\begin{aligned} m_\alpha\, n_\alpha \frac{D\boldsymbol{u}_\alpha}{Dt} = &-\nabla_{\mathrm{r}}\, p_\alpha - \nabla_{\mathrm{r}} \cdot \boldsymbol{\Pi}_\alpha - m_\alpha\, \boldsymbol{u}_\alpha\, (S_\alpha - L_\alpha) + (\rho_{e\alpha}\, \boldsymbol{E} + \boldsymbol{J}_{e\alpha} \times \boldsymbol{B}) \\ &+ \boldsymbol{R}_\alpha \end{aligned} \tag{3.24}$$

where the D/Dt operator is used again. Equation (3.24) is equivalent to the transport equation for macroscopic fluid with electric charge density $\rho_{e\alpha}(\boldsymbol{r}, t) = q_\alpha\, n_\alpha(\boldsymbol{r}, t)$ and current density $\boldsymbol{J}_{e\alpha} = n_\alpha\, q_\alpha\, \boldsymbol{u}_\alpha(\boldsymbol{r}, t)$.

This equation of motion states that the time evolution of fluid momentum for each fluid element of ($\alpha = a, e, i$) particles is due to external electromagnetic forces applied to the fluid, pressure and shear (viscosity) forces. In addition, the collisional interactions $\boldsymbol{R}_\alpha$ as well as the collisional creation and/or recombination of particles through $m_\alpha \boldsymbol{u}_\alpha\, (S_\alpha - L_\alpha)$ also contribute to the momentum.

Specifically, the motions of electrons, ions and neutrals are coupled through the vector $\boldsymbol{R}_\alpha$ which accounts for the momentum exchange in short-range collisions with neutral atoms and also long-range Coulomb encounters between charged particles. Its evaluation requires explicit expressions for $C_\alpha(f_\alpha)$ and the corresponding cross sections for each momentum exchange collision at the microscopic level discussed in volume 1, section 5.4. We postpone this task to section 3.1.3 where a simplified model for $\boldsymbol{R}_\alpha$ essentially valid for short-range collisions will be introduced.

This is also the case of tensor $\boldsymbol{\Pi}_\alpha$ where the calculation of averages $\langle c_i\, c_j \rangle$ in equation (3.14) also involves the velocity distribution function. We face again the problem that the mathematical closure of the macroscopic transport equation (3.24)

[7] For readers who are not familiar with this *convected* time derivative, it is introduced in appendix A.

requires approximations and/or additional physical models, as was indicated in figure 1.1.

Since Boltzmann collision integral is essentially valid for *binary* collisions, the long-range Coulomb interaction between charged particles requires Fokker–Planck or Landau collision operators [2–4].

3.1.3 The friction force

So far we have no expression for the integral of the Boltzmann operator (3.21) which gives $\boldsymbol{R}_\alpha$ in the momentum transport equation. This calculation illustrates the connection between the macroscopic fluid description of plasmas with the properties of elementary processes at the atomic and molecular level discussed in chapters 5 and 6 of volume 1.

The vector $\boldsymbol{R}_\alpha$ accounts for the change in momentum by time and volume units gained or lost by the α-particles in the elastic binary collisions with the other β species,

$$\boldsymbol{R}_\alpha = \sum_{\alpha \neq \beta} \boldsymbol{R}_{\alpha,\beta} \quad \text{where,} \quad \boldsymbol{R}_{\alpha,\beta} = \int_{-\infty}^{+\infty} (m_\alpha\, \boldsymbol{v})\, C_{\alpha,\beta}(f_\alpha, f_\beta)\, d\boldsymbol{v} \tag{3.25}$$

where $C_{\alpha,\beta}(f_\alpha, f_\beta)$ is given by equation (2.16). Equivalently, equation (3.25) represents the momentum gain/loss rate when α-particles are added or removed from the phase space volume $d^3v\, d^3r$ located at point $(\boldsymbol{r},\, t)$ due to collisions with the β species by time unit.

Momentum is conserved in elastic collisions and its change in one particle is equal and opposite to the other colliding particle. Hence, the encounters between α-particles give $\boldsymbol{R}_{\alpha\alpha} = 0$ and are excluded from the summation (3.25) as they do not increase or decrease the average momentum. This is not the case for encounters between different plasma species, for example, a net momentum is transferred from the electron gas to the neutral gas by electron-neutral collisions and then, $\boldsymbol{R}_\alpha \neq 0$. In the general case, the encounters between particles of different species need to be considered in momentum transport equation. Additionally,

$$\boldsymbol{R}_{\alpha\beta} = -\boldsymbol{R}_{\beta\alpha} \quad \text{and,} \quad \sum_\alpha \boldsymbol{R}_\alpha = 0$$

since the amount of momentum gained by one particle group is lost by the others.

In order to derive an expression for $\boldsymbol{R}_{\alpha\beta}$ we consider the elastic collisions between the $dn_\beta = f_\beta(\boldsymbol{v}, \boldsymbol{r}, t)\, d\boldsymbol{v}_\beta$ targets and the $dn_\alpha = f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)\, d\boldsymbol{v}_\alpha$ incident particles. As discussed in sections[8] 5.1 and 5.4 of volume 1, the number of collision events with only *one* β target with the incoming flow $d\Gamma_\alpha = |\boldsymbol{v}_\alpha - \boldsymbol{v}_\beta|\, dn_\alpha$ (see volume 1, equation (5.3)) of α-particles in the time interval δt is,

[8] The distribution function normalization was different in sections 5.1 and 5.4 of volume 1, where $dn = n_o f(\boldsymbol{u})\, d\boldsymbol{u}$, so $\int_{-\infty}^{+\infty} f(\boldsymbol{u})\, d\boldsymbol{u} = 1$, whereas here we use the definition (2.1) usual in kinetic theory and the integral over velocities gives the particle density (2.4).

$$[\,\sigma_{el}(g,\chi)\sin\chi\; d\chi\; d\phi\,] \times \left(\,|v_\alpha - v_\beta|\; dn_\alpha\,\right) \times \delta t$$

Here $\mathbf{g} = \boldsymbol{v}_\alpha - \boldsymbol{v}_\beta$ is the relative velocity and χ the scattering angle (see volume 1, section 5.4 and figure 5.3(b)). The momentum exchanged in one collision along the direction $\boldsymbol{g}$ is given by equation (5.19) in volume 1 as,

$$\Delta\boldsymbol{p} = \mu_{\alpha\beta}\, g\,(1-\cos\chi) \times \frac{\boldsymbol{g}}{g} = \mu_{\alpha\beta}\,(1-\cos\chi)\,\boldsymbol{g}$$

where $\mu_{\alpha\beta} = m_\alpha\, m_\beta(m_\alpha + m_\beta)$ is the reduced mass. The momentum exchanged with *one* β target is the number of collision events with α-particles multiplied by $\Delta\boldsymbol{p}$,

$$\left(\,\mu_{\alpha\beta}\, g\,(1-\cos\chi)\,\boldsymbol{g}\,\right) \times [\,\sigma_{el}(g,\chi)\sin\chi\; d\chi\; d\phi\,] \times \left(\,|v_\alpha - v_\beta|\; dn_\alpha\,\right) \times \delta t$$

When dn_β targets are considered, the momentum exchange rate by time and volume units $\delta\boldsymbol{P}/\delta t$ is given by,

$$\frac{\delta\boldsymbol{P}}{\delta t} = \Delta\boldsymbol{p} \times \dot{Q}_{el} = \left(\mu_{\alpha\beta}\,(1-\cos\chi)\,\boldsymbol{g}\,\right) \times [\,\sigma_{el}(g,\chi)\sin\chi\; d\chi\; d\phi\,] \times \left(|v_\alpha - v_\beta|\; dn_\alpha\,\right) \times dn_\beta$$

Here $\dot{Q}_{el}$ is the *collision rate*, the number of elastic collisions by time unit introduced in chapter 5 of volume 1. Since the scattering angles $(\chi,\, \phi)$ are independent of $\boldsymbol{v}_\alpha$ and $\boldsymbol{v}_\beta$, the integration gives,

$$\frac{\delta\boldsymbol{P}}{\delta t} = \delta\boldsymbol{p} \times \dot{Q}_{el} = \left[2\pi \int_0^\pi \sigma_{el}(g,\chi)\,(1-\cos\chi)\sin\chi\; d\chi\right] \times \mu_{\alpha\beta}\,\boldsymbol{g}\; g\,(f_\alpha\; d\boldsymbol{v}_\alpha)\,(f_\beta\; d\boldsymbol{v}_\beta)$$

The expression between the brackets $\sigma_{tr}(g)$ is the *momentum transfer cross section* for elastic collisions (volume 1, equation (5.20)). Finally, the change in momentum by time and volume units $\boldsymbol{R}_{\alpha\beta}$ is obtained by integration over the velocities $\boldsymbol{v}_\alpha$ and $\boldsymbol{v}_\beta$

$$\boldsymbol{R}_{\alpha\beta} = \mu_{\alpha\beta} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \sigma_{tr}(g)\,\boldsymbol{g}\; g\, f_\alpha\, f_\beta\; d\boldsymbol{v}_\alpha\, d\boldsymbol{v}_\beta$$

This last equation explicitly shows the connection between the physical characteristics of collisions, considered in the momentum transfer cross section $\sigma_{tr}(g)$, and the momentum transport equation (3.24). The electron cross sections of figure 6.3 in volume 1 with neutral gas atoms give the experimental values for $\sigma_{tr}(g)$ that are connected to the macroscopic magnitude $\boldsymbol{R}_{ea}$. This illustrates the influence of the specific properties of collisions at the atomic and molecular level in the macroscopic transport of momentum and energy.

However, the calculation of $\boldsymbol{R}_{\alpha\beta}$ needs, in addition to the cross section $\sigma_{tr}(g)$, the explicit expressions for the velocity distribution functions for the α and β species that are not frequently available. The usual simplifying assumption is to consider equilibrium Maxwellian distributions as in section 3.4 of volume 1 for the calculation of the rate constants. Furthermore, the cross section $\sigma_{tr}(g) \simeq \sigma_{tr}$ is considered constant and independent of the relative velocity. The calculation is cumbersome and it will not be reproduced here, the interested reader can find the details in section 6.5 of reference [3]. The result is,

$$\boldsymbol{R}_{\alpha\beta} = -\mu_{\alpha\beta}\, n_\alpha\, \nu_{\alpha\beta}\, (\boldsymbol{u}_\alpha - \boldsymbol{u}_\beta) \tag{3.26}$$

where $\nu_{\alpha\beta}$ is the collision frequency for momentum transfer given by,

$$\nu_{\alpha\beta} = n_\beta\, \sigma_{tr}\, \frac{4}{3}\left(\frac{8\, k_B\, T}{\pi\, \mu_{\alpha\beta}}\right)$$

In equation (3.26) the collisional interaction between plasma species can be seen as a *friction* or *drag force* $\boldsymbol{R}_{\alpha\beta}$ caused by their relative motion of plasma species.

This expression (3.26) for the friction force is useful for the elastic collisions of electrons ($\mu_{ea} \simeq m_e$) with ions and neutrals. It is also valid for elastic encounters between heavy particles, however, a more involved collision operator is required to account for charge exchange collisions discussed in section 6.3 of volume 1.

3.1.4 The energy transport equation

The next moment of the Boltzmann equation, related to the energy transport is obtained by multiplying equation (2.10) by $m_\alpha v^2/2$ and integrating over $\boldsymbol{v}$ as,

$$\begin{aligned}\frac{\partial}{\partial t}\left(n_\alpha\left\langle\frac{m_\alpha v^2}{2}\right\rangle\right) + \nabla_{\mathbf{r}}\cdot\left(n_\alpha\left\langle\frac{m_\alpha v^2}{2}\,\boldsymbol{v}\right\rangle\right)\\ + \frac{1}{2}\int_{-\infty}^{+\infty} v^2\, \nabla_{\mathbf{v}}\cdot(\boldsymbol{F}_{e\alpha} f_\alpha)\, d\boldsymbol{v} = \int_{-\infty}^{+\infty}\frac{m_\alpha v^2}{2}\, C_s(f_\alpha)\, d\boldsymbol{v}\end{aligned} \tag{3.27}$$

where $\boldsymbol{F}_{e\alpha} = q_\alpha\,[\boldsymbol{E} + (\boldsymbol{v}\times\boldsymbol{B})]$ is again the Lorentz force acting on the α particle. The first term on the left side of equation (3.27) is the time derivative of the energy by volume unit introduced in equation (3.10) as,

$$\frac{\partial}{\partial t}\left(n_\alpha\left\langle\frac{m_\alpha v^2}{2}\right\rangle\right) = \frac{\partial}{\partial t}\left(m_\alpha\, n_\alpha\, \frac{u_\alpha^2}{2} + e_{i\alpha}\right) = \frac{\partial E_\alpha}{\partial t} \tag{3.28}$$

The right term (3.27) involves the collision operator $C_s(f_\alpha)$ as,

$$Q_\alpha = \int_{-\infty}^{+\infty}\frac{m_\alpha v^2}{2}\, C_s(f_\alpha)\, d\boldsymbol{v} \tag{3.29}$$

that will be discussed later and represents the rate of energy density change due to collisions.

The energy flux vector $\boldsymbol{K}_\alpha(\boldsymbol{r}, t)$ of equation 3.18 can be introduced in the second average of equation (3.27) and,

$$\nabla_{\mathbf{r}}\cdot\left(n_\alpha\left\langle\frac{m_\alpha v^2}{2}\,\boldsymbol{v}\right\rangle\right) = \nabla_{\mathbf{r}}\cdot(E_\alpha\,\boldsymbol{u}_\alpha) + \nabla_{\mathbf{r}}\cdot\boldsymbol{q}_\alpha + \nabla_{\mathbf{r}}\cdot(p_\alpha\,\boldsymbol{u}_\alpha) + \nabla_{\mathbf{r}}\cdot(\boldsymbol{\Pi}_\alpha\!:\boldsymbol{u}_\alpha) \tag{3.30}$$

The third term on the left side of equation (3.27) can be transformed as,

$$\nabla_\mathbf{v} \cdot (\boldsymbol{F}_{e\alpha}\, v^2 f_\alpha) = v^2\, \nabla_\mathbf{v} \cdot (\boldsymbol{F}_{e\alpha} f_\alpha) + (f_\alpha\, \boldsymbol{F}_{e\alpha}) \cdot (\nabla_\mathbf{v}\, v^2)$$

Using the divergence theorem we obtain again for a well behaved distribution as $|v| \to \infty$,

$$\int_{-\infty}^{+\infty} v^2\, \nabla_\mathbf{v} \cdot (\boldsymbol{F}_{e\alpha} f_\alpha)\, d\boldsymbol{v} = \underbrace{\int_{-\infty}^{+\infty} \nabla_\mathbf{v} \cdot (v^2 \boldsymbol{F}_{e\alpha} f_\alpha)\, d\boldsymbol{v}}_{\to 0} - \int_{-\infty}^{+\infty} (\boldsymbol{F}_{e\alpha} f_\alpha) \cdot \nabla_\mathbf{v}(v^2)\, d\boldsymbol{v}$$

In the above equation the first integral is null because f_α decreases as $\boldsymbol{v}$ increases and using $\nabla_\mathbf{v}\, (v^2) = 2\, \boldsymbol{v}$ we finally have,

$$\frac{1}{2}\int_{-\infty}^{+\infty} v^2\, \nabla_\mathbf{v} \cdot (\boldsymbol{F}_{e\alpha} f_\alpha)\, d\boldsymbol{v} = -\frac{1}{2}\int_{-\infty}^{+\infty} (\boldsymbol{F}_{e\alpha} f_\alpha) \cdot (2\, \boldsymbol{v})\, d\boldsymbol{v} = -\, n_\alpha\, \langle \boldsymbol{F}_{e\alpha} \cdot \boldsymbol{v} \rangle$$

and the last term in equation (3.27) is the work of the electric field,

$$n_\alpha\, \langle \boldsymbol{F}_{e\alpha} \cdot \boldsymbol{v} \rangle = q_\alpha\, \boldsymbol{E}(\boldsymbol{r},\, t) \cdot \left(n_\alpha \int_{-\infty}^{+\infty} f_\alpha\, \boldsymbol{v}\, d\boldsymbol{v} \right) = \boldsymbol{E} \cdot (q_\alpha\, n_\alpha\, \boldsymbol{u}_\alpha) = \boldsymbol{E} \cdot \boldsymbol{J}_{e\alpha} \qquad (3.31)$$

Finally, substituting in equation (3.27) equations (3.28)–(3.31) we obtain the energy transport equation,

$$\left[\frac{\partial E_\alpha}{\partial t} + \nabla_\mathbf{r} \cdot (E_\alpha\, \boldsymbol{u}_\alpha)\right] = -\, \nabla_\mathbf{r} \cdot \boldsymbol{q}_\alpha - \nabla_\mathbf{r} \cdot (p_\alpha\, \boldsymbol{u}_\alpha) - \nabla_\mathbf{r} \cdot (\boldsymbol{\Pi}_\alpha \colon \boldsymbol{u}_\alpha) + \boldsymbol{J}_{e\alpha} \cdot \boldsymbol{E} + Q_\alpha \qquad (3.32)$$

The left side expresses a continuity equation for the energy E_α in a small fluid element and the right terms are the gain and loss terms. The first right addition is the variation in E_α due to the heat flux $\boldsymbol{q}_\alpha$ and next are the work done by the kinetic pressure $\boldsymbol{P}_\alpha = p_\alpha\, \boldsymbol{I} + \boldsymbol{\Pi}_\alpha$ composed of the scalar pressure p_α and viscosity terms The term $\boldsymbol{J}_{e\alpha} \cdot \boldsymbol{E}$ is the work of current transport in the electric field, whereas Q_α accounts for the collisional energy exchanges.

It is convenient to rewrite equation (3.32) in terms of the pressure p_α or temperature T_α, we expand all terms,

$$\frac{m_\alpha\, u_\alpha^2}{2}\frac{\partial n_\alpha}{\partial t} + m_\alpha\, n_\alpha\, \boldsymbol{u}_\alpha \cdot \frac{\partial \boldsymbol{u}_\alpha}{\partial t} + \frac{\partial e_{i\alpha}}{\partial t} + \nabla_\mathbf{r} \cdot (e_{i\alpha}\, \boldsymbol{u}_\alpha) = -\nabla_\mathbf{r} \cdot (p_\alpha\, \boldsymbol{u}_\alpha)$$
$$-\frac{m_\alpha\, u_\alpha^2}{2}\, \nabla_\mathbf{r} \cdot (n_\alpha\, \boldsymbol{u}_\alpha) - \frac{m_\alpha}{2}(n_\alpha\, \boldsymbol{u}_\alpha) \cdot \nabla_\mathbf{r}\, (u_\alpha^2) - \nabla_\mathbf{r} \cdot \boldsymbol{q}_\alpha - \nabla_\mathbf{r} \cdot (\boldsymbol{\Pi}_\alpha \colon \boldsymbol{u}_\alpha) + \boldsymbol{J}_\alpha \cdot \boldsymbol{E} + Q_\alpha$$

In the second addition in the second line $\boldsymbol{u}_\alpha \cdot \nabla_\mathbf{r}\, (u_\alpha^2) = \boldsymbol{u}_\alpha \cdot [2\, (\boldsymbol{u}_\alpha \cdot \nabla_\mathbf{r})\, \boldsymbol{u}_\alpha]$ and we obtain,

$$\frac{m_\alpha u_\alpha^2}{2}\left[\frac{\partial n_\alpha}{\partial t} + \nabla_\mathbf{r} \cdot (n_\alpha \boldsymbol{u}_\alpha)\right] + \boldsymbol{u}_\alpha \cdot \left[m_\alpha n_\alpha \left(\frac{\partial \boldsymbol{u}_\alpha}{\partial t} + (\boldsymbol{u}_\alpha \cdot \nabla_\mathbf{r})\, \boldsymbol{u}_\alpha\right)\right]$$
$$+ \frac{\partial e_{i\alpha}}{\partial t} + \nabla_\mathbf{r} \cdot (e_{i\alpha} \boldsymbol{u}_\alpha) + \nabla_\mathbf{r} \cdot (p_\alpha \boldsymbol{u}_\alpha) + \nabla_\mathbf{r} \cdot \boldsymbol{q}_\alpha + \nabla_\mathbf{r} \cdot (\boldsymbol{\Pi}_\alpha \colon \boldsymbol{u}_\alpha) = \boldsymbol{J}_\alpha \cdot \boldsymbol{E} + Q_\alpha$$

Between the square brackets are the continuity equation (3.19) and the moment transport equation (3.24) so,

$$\frac{m_\alpha u_\alpha^2}{2}(S_\alpha - L_\alpha)$$
$$+ \boldsymbol{u}_\alpha \cdot \left[-\nabla_\mathbf{r} p_\alpha - \nabla_\mathbf{r} \cdot \boldsymbol{\Pi}_\alpha - m_\alpha \boldsymbol{u}_\alpha (S_\alpha - L_\alpha) + (\rho_{e\alpha} \boldsymbol{E} + \boldsymbol{J}_{e\alpha} \times \boldsymbol{B}) + \boldsymbol{R}_\alpha\right]$$
$$+ \frac{\partial e_{i\alpha}}{\partial t} + \nabla_\mathbf{r} \cdot (e_{i\alpha} \boldsymbol{u}_\alpha) + \nabla_\mathbf{r} \cdot (p_\alpha \boldsymbol{u}_\alpha) + \nabla_\mathbf{r} \cdot \boldsymbol{q}_\alpha + \nabla_\mathbf{r} \cdot (\boldsymbol{\Pi}_\alpha \colon \boldsymbol{u}_\alpha) = \boldsymbol{J}_\alpha \cdot \boldsymbol{E} + Q_\alpha$$

Using the identity $\nabla_\mathbf{r} \cdot (\boldsymbol{\Pi}_\alpha \colon \boldsymbol{u}_\alpha) = \boldsymbol{u}_\alpha \cdot (\nabla_\mathbf{r} \cdot \boldsymbol{\Pi}_\alpha) + (\boldsymbol{\Pi}_\alpha \cdot \nabla_\mathbf{r})\, \boldsymbol{u}_\alpha$ and after a little algebra we obtain,

Missing term

$$\frac{De_{i\alpha}}{Dt} + (e_{i\alpha} + p_\alpha)(\nabla_\mathbf{r} \cdot \boldsymbol{u}_\alpha) + (\boldsymbol{\Pi}_\alpha \cdot \nabla_\mathbf{r})\, \boldsymbol{u}_\alpha + \nabla_\mathbf{r} \cdot \boldsymbol{q}_\alpha$$
$$= Q_\alpha - \boldsymbol{u}_\alpha \cdot \boldsymbol{R}_\alpha + \frac{m_\alpha u_\alpha^2}{2}(S_\alpha - L_\alpha)$$

$+ \vec{J}_\alpha \cdot \vec{E}$

Using the LTE approximation (3.11) this equation can be expressed in terms of the temperature as $e_{i\alpha} = 3n_\alpha k_B T_\alpha/2$ or of the scalar pressure $e_{i\alpha} = 3p_\alpha/2$. In this last case we finally obtain,

Missing term.

$$\frac{3}{2}\frac{Dp_\alpha}{Dt} + \frac{5}{2}p_\alpha (\nabla_\mathbf{r} \cdot \boldsymbol{u}_\alpha) + (\boldsymbol{\Pi}_\alpha \cdot \nabla_\mathbf{r})\, \boldsymbol{u}_\alpha = -\nabla_\mathbf{r} \cdot \boldsymbol{q}_\alpha$$
$$- \boldsymbol{u}_\alpha \cdot \boldsymbol{R}_\alpha + \frac{m_\alpha u_\alpha^2}{2}(S_\alpha - L_\alpha) + Q_\alpha \quad (3.33)$$

$+ \vec{J}_\alpha \cdot \vec{E}$

Here $-\boldsymbol{u}_\alpha \cdot \boldsymbol{R}_\alpha$ is the work of the friction force, the next term accounts for the kinetic energy of created/destroyed α-particles. The term Q_α in equation (3.34c) is the second order moment (3.29),

$$Q_\alpha = \int_{-\infty}^{+\infty} \frac{m_\alpha v^2}{2}\, C_s(f_\alpha)\, dv$$

and again, its rigorous calculation is cumbersome so we will make use of approximate expressions. The term Q_α accounts for the energy transformed in inelastic collisions. For example, the ionizing electron loses an amount of energy E_I equal to the ionization potential of the neutral gas and when the new ion is created at rest with respect to the ion fluid we can write,

$$Q_I = E_i\, \nu_I$$

where ν_i is the ionization frequency and is given by, $\nu_I \, n_e = k_I \, n_a \, n_e$ introduced in chapter 6 of volume 1.

Equation (3.33) shows again the connection between the elementary processes at the molecular and atomic level with the macroscopic description of a plasma. Finally, the heat flux vector $\boldsymbol{q}_\alpha$, the collisional energy exchange Q_α between plasma species and the components of the tensor $\boldsymbol{P}_\alpha = p_\alpha \, \boldsymbol{I} + \boldsymbol{\Pi}_\alpha$ still remain undetermined in the above energy transport equation. Again, their calculation requires an expression of the velocity distribution function $f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)$ and/or additional approximations. These undetermined quantities are also related to the *closure problem* introduced in section 3.2.1.

3.2 The hydrodynamic plasma transport equations

The above fluid transport equations (3.19), (3.24) and (3.32) for a plasma ($\alpha = e, i, a$) are,

$$\frac{\partial n_\alpha}{\partial t} + \nabla_{\mathbf{r}} \cdot (n_\alpha \, \boldsymbol{u}_\alpha) = S_\alpha - L_\alpha \tag{3.34a}$$

$$m_\alpha \, n_\alpha \frac{D\boldsymbol{u}_\alpha}{Dt} = -\nabla_{\mathbf{r}} \, p_\alpha - \nabla_{\mathbf{r}} \cdot \boldsymbol{\Pi}_\alpha - m_\alpha \, \boldsymbol{u}_\alpha \, (S_\alpha - L_\alpha) + \boldsymbol{F}_{e\alpha} + \boldsymbol{R}_\alpha \tag{3.34b}$$

$$\begin{aligned} &\frac{3}{2}\frac{Dp_\alpha}{Dt} + \frac{5}{2} p_\alpha \, (\nabla_{\mathbf{r}} \cdot \boldsymbol{u}_\alpha) + (\boldsymbol{\Pi}_\alpha \cdot \nabla_{\mathbf{r}}) \, \boldsymbol{u}_\alpha \\ &\quad + \nabla_{\mathbf{r}} \cdot \boldsymbol{q}_\alpha = Q_\alpha - \boldsymbol{u}_\alpha \cdot \boldsymbol{R}_\alpha + \frac{m_\alpha \, u_\alpha^2}{2} (S_\alpha - L_\alpha) \end{aligned} \tag{3.34c}$$

Missing term $+ \vec{J}_\alpha \cdot \vec{E}$

where $\boldsymbol{F}_{e\alpha} = (\rho_{e\alpha} \, \boldsymbol{E} + \boldsymbol{J}_{e\alpha} \times \boldsymbol{B})$ with $\rho_{e\alpha} = q_\alpha \, n_\alpha$ and $\boldsymbol{j}_{e\alpha} = n_\alpha \, q_\alpha \, \boldsymbol{u}_\alpha$ are the electromagnetic forces. The vector $\boldsymbol{R}_\alpha$ is the sum of the *friction forces* $\boldsymbol{R}_\alpha = \sum_{\alpha \neq \beta} \boldsymbol{R}_{\alpha\beta}$ introduced in section 3.1.3. In the equilibrium of particle production and losses $S_\alpha = L_\alpha$ and some terms in equations (3.34) are null.

Equations (3.34) are also called *multifluid plasma equations* and have been derived from the Boltzmann equation (2.10) under the assumption of the LTE *local equilibrium* introduced in section 4.2 of volume 1. These are collision-dominated plasmas where fast energy relaxation processes relax fluctuations on time and length $l_c \ll L_s$ and time $\tau \ll L_s/|\boldsymbol{u}_\alpha|$ scales much shorter than the macroscopic plasma motion L_s as discussed in the introductory section.

The plasma medium can be also considered as a *conducting fluid* and equations (3.34) can be combined to describe its macroscopic behavior *as a whole* without regarding the individual motions of its constituent species. The parameters of interest are such as the total mass density as $\rho_m = \sum_\alpha \rho_{m\,\alpha}$ or the mean fluid velocity $\rho_m \, \boldsymbol{u} = \sum_\alpha \rho_{m\,\alpha} \, \boldsymbol{u}_\alpha$ where the contribution of each plasma species is weighted. In order to derive the hydrodynamic equations for an electrically conductive single fluid, the previous formulation of transport equations needs to be modified accordingly.

The resulting transport equations for a conducting fluid are called the *magnetohydrodynamic equations* (MHD) and are not frequently used in their complex general form. Several approximations are usually considered to eliminate some terms and to reduce the MHD equations to a simplified equation set. However, these tasks are beyond the scope of the present book.

3.2.1 The closure of the transport equations

Unfortunately, the equations (3.34) have undetermined coefficients related with the moments of the distribution function $f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)$ that need to be supplied by additional physical models and/or approximations.

The first moment of the Boltzmann equation (2.10) gives the continuity equation (3.34a) that relates the number density $n_\alpha(\boldsymbol{r}, t)$ with the fluid velocity $\boldsymbol{u}_\alpha(\boldsymbol{r}, t)$. Since these are two independent macroscopic variables we need an additional moment to derive the momentum conservation equation (3.34b). However, this relates the fluid velocity with the kinetic stress tensor $\boldsymbol{P}_\alpha = p_\alpha \boldsymbol{I} + \boldsymbol{\Pi}_\alpha$ and equations (3.34a) and (3.34b) become two equations with three independent macroscopic variables. The following moment gives the energy transport equation (3.34c) which involves the heat flux vector $\boldsymbol{q}(\boldsymbol{r}, t)$ in addition to kinetic pressure $\boldsymbol{S}_\alpha$, the number density $n_\alpha(\boldsymbol{r}, t)$ and the mean velocity $\boldsymbol{u}_\alpha(\boldsymbol{r}, t)$.

Consequently, any *finite* number of moments of the Boltzmann equation (2.10) lead a set of fluid equations with more unknowns than equations. This problem of the *mathematical closure* is inherent to macroscopic transport equations and we restrict ourselves to the first three moments since these are related with the magnitudes of physical interest.

Conventional hydrodynamics circumvents this problem introducing phenomenological expressions for the undetermined variables in equation (3.34), For example, the usual expression for the stress tensor $\boldsymbol{\Pi}_\alpha$ or the Fourier law $\boldsymbol{q} = -\kappa \nabla_{\mathbf{r}} T$ for the heat flux where T is the local temperature and κ is the thermal conductivity[9] of gases or liquids. These relations constitute the *additional information* mentioned in the scheme of figure 1.1.

However, this procedure is not fully justified in the case of plasmas where similar expressions are not available. The explicit form of the velocity distribution function is required—or equivalently—a specific *plasma model* is required to find expressions for the undetermined variables in the fluid equations. For example, the evaluation of the heat flux vector,

$$\boldsymbol{q}_\alpha = \int_{-\infty}^{+\infty} \left(\frac{m_\alpha \boldsymbol{c}^2}{2}\right) \boldsymbol{c}\, f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)\, d\boldsymbol{c}$$

requires a specific velocity distribution $f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t)$ (or equivalently, a physical model) for electrons, ions and neutral atoms. The same argument applies to the other indeterminate quantities in equations (3.34).

There are two available schemes to circumvent this difficulty. The simplest choice is to *truncate* the moment expansion, assuming the higher order moments arbitrarily as zero or alternatively, in terms of lower order moments on the basis of physical

[9] In the general case the thermal conductivity is also a tensor.

assumptions. The implicit risk is not to reflect relevant physical phenomena in these simple approaches, such as the *cold* and *warm* plasma models discussed below.

The alternative is *asymptotic schemes* where the mathematical expansion of a small parameter controls the approximations, as for example, the ratio $\epsilon = l_c/L_c \ll 1$ in Chapman–Enskog theory [2, 4] for low pressure neutral gases. This permits expanding the distribution function as,

$$f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t) = f_{o,\alpha}(\boldsymbol{v}, \boldsymbol{r}, t) + f_{1,\alpha}(\boldsymbol{v}, \boldsymbol{r}, t)\,\epsilon + f_{2,\alpha}(\boldsymbol{v}, \boldsymbol{r}, t)\,\epsilon^2 \ldots$$

In this expansion the functions $f_{i,\alpha}(\boldsymbol{v}, \boldsymbol{r}, t)$ with $i = 1, 2, \ldots$ are assumed of the same order of magnitude and reference LTE equilibrium state $f_{o,\alpha}(\boldsymbol{v}, \boldsymbol{r}, t)$ is described by a local Maxwellian distribution (equation (4.3) of volume 1). This asymptotic closure scheme based on the $\epsilon = l_c/L_c \ll 1$ parameter is valid for collisional plasmas where short-range encounters dominate.

The second approach for magnetized plasmas is based on the ratio $\epsilon = R_{l\alpha}/L_c \ll 1$ between the Larmor radius $R_{l\alpha}$ (see section 4.5.4 of volume 1) and the macroscopic scale of plasma motion. The resulting macroscopic equations valid for fully ionized and magnetized plasma flows are called *Braginskii equations* [2, 5]. No asymptotic closure scheme for collisionless unmagnetized plasmas has been formulated so far.

These asymptotic expansion closure approaches are cumbersome and less employed than the truncation schemes discussed below. The interested reader is referred to the specialized literature [2, 4, 5] since this specific subject is outside the scope of the present work.

3.2.2 The cold and warm plasma models

The simplest closed set of hydrodynamic plasma equations is called the *cold plasma* model where the kinetic pressure $\boldsymbol{P}_\alpha = p_\alpha\,\boldsymbol{I} + \boldsymbol{\Pi}_\alpha$ is neglected. Equations (3.34) reduce in this case to,

$$\frac{\partial n_\alpha}{\partial t} + \nabla_{\mathbf{r}} \cdot (n_\alpha\,\boldsymbol{u}_\alpha) = S_\alpha - L_\alpha \tag{3.35a}$$

$$m_\alpha\,n_\alpha\,\frac{D\boldsymbol{u}_\alpha}{Dt} = (\rho_{e\alpha}\,\boldsymbol{E} + \boldsymbol{J}_{e\alpha} \times \boldsymbol{B}) + \boldsymbol{R}_\alpha - m_\alpha\,\boldsymbol{u}_\alpha\,(S_\alpha - L_\alpha) \tag{3.35b}$$

In the *cold plasma* approximation the thermal motions of particles are disregarded or equivalently zero kinetic temperatures T_α are considered for the plasma species. Hence, this model considers their velocity distribution functions,

$$f_\alpha(\boldsymbol{v}, \boldsymbol{r}, t) = \delta[\boldsymbol{v} - \boldsymbol{u}_\alpha(\boldsymbol{r}, t)]$$

are Dirac deltas centered at the macroscopic flow velocity.

The conservation of energy is the simplifying approximation introduced in the *warm plasma* model. The terms involving the heat flux vector $\boldsymbol{q}$ or equivalently, the plasma thermal conductivity are considered as null in equations (3.34). Hence the plasma is non-viscous and also the tensor $\boldsymbol{\Pi}_\alpha$ corresponding to the non-diagonal terms of the kinetic pressure are neglected and we have,

$$\frac{\partial n_\alpha}{\partial t} + \nabla_{\mathbf{r}} \cdot (n_\alpha\, \boldsymbol{u}_\alpha) = S_\alpha - L_\alpha \tag{3.36a}$$

$$m_\alpha\, n_\alpha\, \frac{D\boldsymbol{u}_\alpha}{Dt} = -\nabla_{\mathbf{r}}\, p_\alpha + (\rho_{e\alpha}\, \boldsymbol{E} + \boldsymbol{J}_{e\alpha} \times \boldsymbol{B}) + \boldsymbol{R}_\alpha - m_\alpha\, \boldsymbol{u}_\alpha\, (S_\alpha - L_\alpha) \tag{3.36b}$$

$$\frac{3}{2}\, \frac{Dp_\alpha}{Dt} + \frac{5}{2}\, p_\alpha\, (\nabla_{\mathbf{r}} \cdot \boldsymbol{u}_\alpha) = Q_\alpha - \boldsymbol{u}_\alpha \cdot \boldsymbol{R}_\alpha + \frac{m_\alpha\, u_\alpha^2}{2}\, (S_\alpha - L_\alpha) \tag{3.36c}$$

+ $\vec{J}_\alpha \cdot \vec{E}$ missing term

In this approximation, the scalar pressure p_α is preserved to be consistent with the LTE equilibrium conditions as it is proportional to the kinetic temperature T_α of the plasma species (see equations (3.12) and (3.13)).

The equations for a plasma in equilibrium derived in chapter 4 of volume 1 on the basis of the Maxwellian velocity distribution function are recovered from equation (3.36b) as shown in box 3.2. Furthermore, when collisions are neglected, $\boldsymbol{R}_\alpha$, Q_α are null as well as L_α and S_α. In this case the warm plasma model is also denominated *adiabatic approximation* since equation (3.36c) reduces to the $p_\alpha\, \rho_{m\alpha}^{-\gamma}$= constant, as shown in box 3.3.

Finally, *hybrid* approaches are possible where ions and/or neutrals are considered *cold* since their typical kinetic temperatures are much lower than those of the electron fluid as in section 3.4. These can be considered *warm* compared with the ion fluid background since the thermal motion of the lighter electrons is much faster.

Box 3.2. Plasma equilibrium

In the steady equilibrium of an electron–ion ($\alpha = i$, e and $q_\alpha = \pm e$) plasma under an external electric field $\boldsymbol{E} = -\nabla_{\mathbf{r}}\phi$ with $\boldsymbol{B} = 0$ and equation (3.36b) becomes,

$$m_\alpha\, n_\alpha\, \cancelto{0}{\frac{D\boldsymbol{u}_\alpha}{Dt}} = -\nabla_{\mathbf{r}}\, p_\alpha + \rho_{e\alpha}\, \boldsymbol{E} = 0 \quad \text{and,} \quad -\nabla_{\mathbf{r}}\, (n_\alpha\, k_B\, T\,) \pm n_\alpha\, e\, (-\nabla_{\mathbf{r}}\phi) = 0$$

The fluid velocity is $\boldsymbol{u}_\alpha = 0$ since the plasma remains at rest and $S_\alpha = L_\alpha$ as the ionization and recombination rates of charged particles are equal. In these conditions,

$$\nabla_{\mathbf{r}}\left[\ln(n_\alpha) \mp \frac{e\,\phi}{k_B\, T}\right] = 0 \quad \text{hence,} \quad \ln(n_\alpha) \mp \frac{e\,\phi}{k_B\, T} = \text{constant}$$

The expression between the brackets is uniform and we recover equations (4.5) and (4.5) of volume 1 for a plasma in equilibrium,

$$n_\alpha = n_o\, \exp\left(\mp \frac{e\,\phi}{k_B T}\right)$$

These expressions were derived in chapter 4 of volume 1 using the Maxwellian distribution function (4.4) for the velocity distribution of ions and electrons in the Maxwellian plasma equilibrium state.

Box 3.3. The adiabatic approximation

When collisions can be neglected equations (3.36a) and (3.36c) reduce to,

$$\frac{\partial n_\alpha}{\partial t} + \nabla_{\mathbf{r}} \cdot (n_\alpha\, \boldsymbol{u}_\alpha) = 0$$
$$\frac{3}{2}\frac{Dp_\alpha}{Dt} + \frac{5}{2}\, p_\alpha\, (\nabla_{\mathbf{r}} \cdot \boldsymbol{u}_\alpha) = 0$$

We can use the first equation to substitute $\nabla_{\mathbf{r}} \cdot \boldsymbol{u}_\alpha$ in the second since,

$$\frac{\partial n_\alpha}{\partial t} + \boldsymbol{u}_\alpha \cdot \nabla_{\mathbf{r}} n_\alpha + n_\alpha \nabla_{\mathbf{r}} \cdot \boldsymbol{u}_\alpha = \frac{Dn_\alpha}{Dt} + n_\alpha \nabla_{\mathbf{r}} \cdot \boldsymbol{u}_\alpha = 0$$

We multiply this last equation by the particle mass m_α so $m_\alpha\, n_\alpha = \rho_{m\alpha}$ is the density, then,

$$\frac{3}{2}\frac{1}{p_\alpha}\frac{Dp_\alpha}{Dt} - \frac{5}{2}\frac{1}{\rho_{m\alpha}}\frac{D\rho_{m\alpha}}{Dt} = 0, \qquad \frac{D}{Dt}\left[\frac{3}{2}\ln(p_\alpha) - \frac{5}{2}\ln(\rho_{m\alpha})\right] = 0$$

Finally we obtain the relation $p_\alpha\, \rho_{m\alpha}^{-5/3}$ = constant usual in thermodynamics for the adiabatic expansion of gas with $\gamma = 5/3$, since we made use of the ideal monoatomic gas approximation (3.11).

This applies to PLE plasmas discussed in sections 4.2 and 4.3 of volume 1 so far as the macroscopic local equilibrium is preserved on some length and time scales.

3.2.3 The diffusion approximation

In the steady equilibrium of unmagnetized ($\boldsymbol{B} = 0$) weakly ionized plasmas the production and losses of charged particles are balanced and short-range collisions with neutral atoms background are dominant. We consider a *cold* (T_i, $T_e \gg T_a$) and uniform distribution of neutrals with a macroscopic velocity ($\boldsymbol{u}_a \simeq 0$) much slower than those of charged particles.

In these conditions the friction force (3.26) for ions and electrons can be approximated by $\boldsymbol{R}_{\alpha a} = -\mu_{\alpha a}\, n_\alpha\, \nu_{\alpha a}\, \boldsymbol{u}_\alpha$ since long-range Coulomb collisions between charged particles are negligible. The momentum transport equation (3.36b) for ions and electrons becomes,

$$m_\alpha\, n_\alpha \frac{D\boldsymbol{u}_\alpha}{Dt} = -\nabla_{\mathbf{r}}\,(n_\alpha\, k_B\, T) \pm e\, n_\alpha\, \boldsymbol{E} - \mu_{\alpha a}\, n_\alpha\, \nu_{\alpha a}\, \boldsymbol{u}_\alpha$$

where $\mu_{ia} = m_i/2$ and $\mu_{ea} = m_e$ and $\nu_{\alpha a}$ are the corresponding momentum transfer collision frequencies. When the macroscopic acceleration $D\boldsymbol{u}_\alpha/Dt \simeq 0$ is null the force imparted on charged particles by the electric field is balanced by collisions and pressure forces. Then,

$$\boldsymbol{u}_\alpha = \pm \frac{e}{\mu_{\alpha a}\,\nu_{\alpha a}}\boldsymbol{E} - \frac{k_B\,T}{\mu_{\alpha a}\,\nu_{\alpha a}}\,\frac{\nabla_{\mathbf{r}}\,n_\alpha}{n_\alpha} \tag{3.37}$$

the velocity of ions and electrons has two components; one along the gradient $\nabla_{\mathbf{r}}\,n_\alpha$ and a *drift speed* $\boldsymbol{u}_{d\alpha} = \mu_\alpha\,\boldsymbol{E}$ parallel to the electric field $\boldsymbol{E}$. The coefficient $\mu_\alpha = (e/\mu_{\alpha a}\,\nu_{\alpha a})$ where $\mu_{\alpha,a} = m_a\,m_\alpha/(m_a + m_\alpha)$ is called *mobility* and the second $D_\alpha = (k_B\,T/\mu_{\alpha a}\,\nu_{\alpha a})$ is a *diffusion coefficient*. We can write for the particle flux,

$$\boldsymbol{\Gamma}_\alpha = n_\alpha\,\boldsymbol{u}_\alpha = \pm\,\mu_\alpha\,n_\alpha\,\boldsymbol{E} - D_\alpha\,\nabla_{\mathbf{r}}\,n_\alpha$$

which reduces to the Fick's law of diffusion when $\boldsymbol{E} = 0$ or also for neutral particles, which are unaffected by the electric field.

This equation expresses that in isothermal plasmas, short-range collision dominating the motion of ions and electrons is driven by the electric field with velocity $\boldsymbol{v}_{d\alpha}$ and the charges also perform a random walk displacement from regions with high concentrations to less dense zones. Since the electric current transported is $\boldsymbol{J}_e = \sigma_c\,\boldsymbol{E}$ the drift speed of charged species ($\alpha = e,\,i$) contributes to the plasma electric conductivity σ_c.

However, both charged species cannot move separately so $\boldsymbol{\Gamma}_i = \boldsymbol{\Gamma}_e$ and quasineutrality $n_i \simeq n_e = n$ must be preserved, otherwise intense electric fields develop in the plasma. Thus, we can write,

$$\boldsymbol{\Gamma}_i = \mu_i\,n\,\boldsymbol{E} - D_i\,\nabla_{\mathbf{r}}\,n = \mu_e\,n\,\boldsymbol{E} - D_e\,\nabla_{\mathbf{r}}\,n = \boldsymbol{\Gamma}_e$$

Since the electric field is the same for both charged species we have,

$$\boldsymbol{E}_a = \frac{D_i - D_e}{\mu_i + \mu_e}\,\frac{\nabla_{\mathbf{r}}\,n}{n}$$

and substituting in $\boldsymbol{\Gamma}_i$ we finally obtain,

$$\boldsymbol{\Gamma}_i = -\left(\frac{\mu_i\,D_e + \mu_e\,D_i}{\mu_i + \mu_e}\right)\nabla_{\mathbf{r}}\,n = -\,D_a\,\nabla_{\mathbf{r}}\,n$$

where D_a is called *ambipolar diffusion coefficient*[10]. The last expression states that the flow of massive ions is affected by the mobility μ_e and free diffusion D_e coefficients of electrons through the *ambipolar* electric field $\boldsymbol{E}_a$ calculated above. Since electrons and ions interact through the electric field, their motions are not independent to preserve the plasma quasineutrality.

3.3 Electron and ion waves

The short-range collisions between neutral atoms produce in a gas or liquid zones where particles are compressed together or are spread apart. These local low amplitude pressure and/or density fluctuations produce a sound wave that

[10] The term *ambipolar diffusion* has a different meaning in astrophysics where it refers to the decoupling process of neutral atoms from the plasma during the initial stages of star formation.

propagates in the gas. The situation is more complex in a plasma since it contains ions and electrons that also interact through the plasma electric field. The motion of heavy ions pulls off the lighter electrons to preserve the plasma quasineutrality. Consequently, there are oscillations and waves of plasma charged particles even in the absence of long and short range collisions.

Next, we will derive the expression for the sound velocity in a neutral gas, since this analysis will be useful to study the oscillations of ions and electrons. As we will see, the disparity of the ion and electron movement time scales introduces different plasma waves and oscillations.

We recall first a few concepts[11] before starting, the equation,

$$\boldsymbol{H}(\boldsymbol{r},\, t) = \boldsymbol{H}_o \exp[i\ (\boldsymbol{k} \cdot \boldsymbol{r} - \omega\ t)]$$

describes a physical magnitude with *amplitude* $\boldsymbol{H}_o$ periodic in time and space. Along the direction pointed by the *wavenumber vector* $\boldsymbol{k}$ this wave transports energy.

The *wavelength* λ is the distance between two points with equal amplitude (or phase) and $|\boldsymbol{k}| = 2\pi/\lambda$ gives the propagation rate in space. The *period* T is the time elapsed between two points with equal amplitude (or phase) and the *angular frequency* $\omega = 2\pi/T$ characterizes the oscillation in time.

The *phase velocity* $v_f = \lambda/T$ or equivalently $v_f = \omega/|\boldsymbol{k}|$ is the rate at which a point of the wave with constant phase moves along the direction where $\boldsymbol{k}$ points. However, the variation in wave amplitude or *envelope* propagates in space at the *group velocity* $v_g = d\omega/dk$ and $\omega(\boldsymbol{k})$ is called the *dispersion function*. Both speeds v_f and v_g are equal only when $\omega(\boldsymbol{k})$ is a linear function.

3.3.1 Neutral gas sound waves

Neglecting viscosity effects, for the conservation of mass and momentum for ordinary gas of density ρ and velocity $\boldsymbol{u}$ of a neutral gas we have,

$$\frac{\partial \rho}{\partial t} + \nabla_{\mathbf{r}} \cdot (\rho\ \boldsymbol{u}) = 0 \qquad (3.38a)$$

$$\rho \left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla_{\mathbf{r}})\boldsymbol{u} \right] = -\nabla_{\mathbf{r}}\, p_a \qquad (3.38b)$$

The equations of motion (3.38) are closed using the thermodynamic relation $p_a\, \rho^{-\gamma} = C$ where C is a constant and $\gamma = C_p/C_v$ is the adiabatic gas constant. This equation expresses that local gas rarefaction and compression are considered as *adiabatic* processes. That is, the molecular collision time scales is very fast and the gas rapidly relaxes to a local equilibrium. We can write,

$$\nabla_{\mathbf{r}}\, p_a = C\ \gamma\ \rho^{(\gamma-1)}\ \nabla_{\mathbf{r}}\rho \quad \text{and then,} \quad \nabla_{\mathbf{r}}\, p_a = \gamma\ p_a\ \frac{\nabla_{\mathbf{r}}\, \rho}{\rho} \qquad (3.39)$$

that closes the above equations. We consider the small amplitude oscillations of physical magnitudes ($\boldsymbol{u}$, p_a, ρ) with respect to a fixed equilibrium state of the gas at

[11] See sections 4.1 and 4.2 of reference [6].

rest $\boldsymbol{u}_o \simeq 0$ where gas pressure and density (p_{oa}, ρ_o) are uniform and constant in time. For low amplitude oscillations we can approximate

$$\boldsymbol{u} \simeq \boldsymbol{u}_1\, e^{i\,(\boldsymbol{k}\cdot\boldsymbol{r}-\omega\, t)}, \quad p_a \simeq p_{ao} + p_{a1}\, e^{i\,(\boldsymbol{k}\cdot\boldsymbol{r}-\omega\, t)} \quad \text{and also,} \quad \rho \simeq \rho_o + \rho_1\, e^{i\,(\boldsymbol{k}\cdot\boldsymbol{r}-\omega\, t)}$$

where ω is the frequency of the oscillation, $\boldsymbol{k}$ is the wave number vector with $|\boldsymbol{k}| = 2\,\pi/\lambda$ where λ is the perturbation wavelength. Neglecting the contribution of second order terms such as $(\boldsymbol{u}_1 \times \rho_1)$, $(p_1 \times \rho_1)$, etc, equations (3.38b) and (3.38a) are linearized. Thus, $(\boldsymbol{u} \cdot \nabla_{\mathbf{r}})\,\boldsymbol{u}$ in equation (3.38b) is of second order and it is neglected, for the pressure gradient (3.39),

$$\nabla_{\mathbf{r}} p_a \sim \gamma\, \frac{p_{ao} + p_{a1}\, e^{i\,(\boldsymbol{k}\cdot\boldsymbol{r}-\omega\, t)}}{\rho_o + \rho_1\, e^{i\,(\boldsymbol{k}\cdot\boldsymbol{r}-\omega\, t)}} \times (i\,\boldsymbol{k})\rho_1\, e^{i\,(\boldsymbol{k}\cdot\boldsymbol{r}-\omega\, t)}$$

and neglecting second order contributions,

$$\nabla_{\mathbf{r}}\, p_a \sim \gamma\,(i\,\boldsymbol{k})\, e^{i\,(-)} \times \frac{\rho_1\, p_o}{\rho_o + \rho_1\, e^{i\,(-)}} \simeq \gamma\,(i\,\boldsymbol{k})\, e^{i\,(-)} \times \frac{p_{ao}\,\rho_1}{\rho_o} \times \underbrace{\frac{1}{1 + e^{i(-)}\,\rho_1/\rho_o}}_{\sim 1}$$

The linearized equations (3.38b) and (3.38a) are,

$$-\omega\,\rho_1 + (\boldsymbol{k}\cdot\boldsymbol{u}_1)\,\rho_o = 0 \tag{3.40}$$

$$-\omega\,\rho_o\,\boldsymbol{u}_1 + \gamma\,\frac{p_o}{\rho_o}\,\rho_1\,\boldsymbol{k} = 0 \tag{3.41}$$

Taking the scalar product with $\boldsymbol{k}$ of equation (3.41) and eliminating the perturbation amplitudes $(\boldsymbol{u}_1 \cdot \boldsymbol{k})$ and ρ_1 is derived the dispersion relation $\omega = c_s\, k$ as,

$$\frac{\omega}{k} = \sqrt{\frac{\gamma\, k_B\, T}{m_a}} \tag{3.42}$$

The pressure and density fluctuations propagate at the sound velocity c_s in a neutral gas and group velocity is equal to phase velocity. The propagation velocity of perturbations depends on the thermal energy and collisions between neutral gas molecules transport pressure waves.

3.3.2 Ion waves

Now we turn our attention to the ion motion that has a slower time scale compared with the electron response. Neglecting collisions, the ion number conservation equation (3.36a) and momentum (3.36b) for ions are,

$$\frac{\partial n_i}{\partial t} + \nabla_{\mathbf{r}} \cdot (n_i\,\boldsymbol{u}_i) = 0 \tag{3.43a}$$

$$m_i\, n_i \left[\frac{\partial \boldsymbol{u}_i}{\partial t} + (\boldsymbol{u}_i \cdot \nabla_{\mathbf{r}})\boldsymbol{u}_i\right] = e\, n_e\,\boldsymbol{E} - \nabla_{\mathbf{r}}\, p_i \tag{3.43b}$$

We will also make use of the *adiabatic approximation* (3.39) $\nabla_{\mathbf{r}}\, p_i = \gamma\,(k_B\, T_i)\, \nabla_{\mathbf{r}}\, n_i$ and the field is $\boldsymbol{E} = -\nabla_{\mathbf{r}}\, \phi$ where ϕ is the plasma electric potential.

The contribution of electrons to the momentum fluctuations can be neglected and its density n_e can be related with the perturbed plasma potential $\phi = \phi_1 \exp[i\,(\boldsymbol{k} \cdot \boldsymbol{r} - \omega\, t)]$ using equation (4.6) of volume 1,

$$n_e(\phi) = n_{eo} \exp\left(\frac{e\,\phi_1}{k_B\, T_e}\right) = n_{eo}\left[1 + \frac{e\,\phi_1}{k_B\, T_e} + \dots\right] \sim n_{eo} + n_{e1}$$

with $n_{e1} = n_{eo}\,(e\,\phi_1/k_B\, T_e)$. In the equilibrium state $n_{io} = n_{eo} = n_o$ and $n_{i1} = n_{e1}$ since slow changes in the ion density are rapidly followed by the fast electrons.

As in previous sections 3.3.3 and 3.3.1, equations (3.43) can be linearized and we have,

$$\begin{aligned} -\,\omega\, n_{i1} + \; n_o\,(\boldsymbol{u}_1 \cdot \boldsymbol{k}) &= 0 \\ -\,\omega\, n_{io}\, m_i\, \boldsymbol{u}_{i1} + e\, n_{io}\, \phi_1\, \boldsymbol{k} + (k_B\, T_i)\,\gamma\, n_{i1}\, \boldsymbol{k} &= 0 \\ n_{i1} - n_o\, \frac{e\,\phi_1}{k_B\, T_e} &= 0 \end{aligned} \tag{3.44}$$

Eliminating the perturbation amplitudes n_{e11}, $(\boldsymbol{k} \cdot \boldsymbol{u}_{i1})$ and ϕ_1 as before,

$$\omega^2 = k^2 \left[\frac{k_B(\gamma\, T_i + T_e)}{m_i}\right]$$

and the dispersion relation is $\omega = c_{is}\, k$ where,

$$c_{is} = \sqrt{\frac{k_B(\gamma\, T_i + T_e)}{m_i}}$$

is the *ion sound speed.* As $c_{is} = \omega/k$ is constant *ion sound* or *ion acoustic waves* propagate without dispersion with identical group and phase wave speeds.

The fluctuations of ions density are similar to sound in neutral gases, the compression and rarefaction of ion density pull the electrons along to shield the fluctuations of the electric field. Ion waves only exist when ions have thermal motion, since c_{is} depends on T_e a high average energy of electrons makes the electric field shielding process faster. The propagation of ion sound waves is independent of ion collisions, contrary to ordinary sound waves in neutral gases.

In PLE plasmas the ion temperature T_i is much lower than $T_e \gg T_i$ thermal energy of electrons is dominant and $c_{is} \simeq \sqrt{k_B T_e/m_i}$. Using the ion plasma frequency $\omega_{pi} = \sqrt{e^2\, n_o/m_i\, \epsilon_o}$ we can write,

$$\frac{\omega^2}{\omega_{pi}^2} \simeq \left(\frac{m_i\, \epsilon_o}{e^2\, n_o}\right) \times \left(\frac{k_B\, T_e}{m_i}\right) = k^2\, \lambda_D^2 = 4\pi^2\, \frac{\lambda_D^2}{\lambda^2}$$

where λ represents the wavelength of the perturbation.

The maximum frequency of ion waves is $\omega \sim \omega_{pi}$ corresponding to the shortest perturbation wavelengths $\lambda \sim \lambda_D$ associated with the minimum plasma scale length, in the order of the Debye length (maximum $k = 2\pi/\lambda$). The wave frequency ω decreases for fluctuation wavelengths $\lambda \gg \lambda_D$ (decreasing $k = 2\pi/\lambda$).

3.3.3 Electron waves

We start with the warm plasma model introduced in section 3.2.2. Neglecting collisions, we can drop in equation (3.36) the friction forces, collisional ionization, recombination terms, etc. The continuity (3.36a) and momentum transport (3.36b) for electrons are,

$$\frac{\partial n_e}{\partial t} + \nabla_{\mathbf{r}} \cdot (n_e \, \boldsymbol{u}_e) = 0 \tag{3.45a}$$

$$m_e \, n_e \left[\frac{\partial \boldsymbol{u}_e}{\partial t} + (\boldsymbol{u}_e \cdot \nabla_{\mathbf{r}})\boldsymbol{u}_e \right] = -e \, n_e \, \boldsymbol{E} - \nabla_{\mathbf{r}} \, p_e \tag{3.45b}$$

These are similar to equation (3.38) for a neutral gas except by the plasma electric field $\boldsymbol{E}$ in equation (3.45b). In addition to the collisionless *adiabatic approximation* where $p_e \, n_e^{-\gamma} = cte.$ (see box 3.3) we will need further assumptions involving the electric field to close this system of equations.

First we will consider the fast time scale for the electric field fluctuations that only electrons can follow. In this case, the ions are considered a uniform background fixed in time with uniform density n_{io} and only the electron density $n_e = n_{eo} + n_{e1} \exp(i \, (\boldsymbol{k} \cdot \boldsymbol{r} - \omega \, t))$ fluctuates and $n_{eo} = n_{io} = n_o$ in the state of equilibrium as fast electrons preserve the plasma quasineutrality. Introducing in the Poisson equation $\nabla_{\mathbf{r}} \cdot \boldsymbol{E} = e(n_i - n_e)/\epsilon_o$ the electric field $\boldsymbol{E} = \boldsymbol{E}_1 \exp(i \, (\boldsymbol{k} \cdot \boldsymbol{r} - \omega \, t))$,

$$\nabla_{\mathbf{r}} \cdot \boldsymbol{E} = \frac{e}{\epsilon_o}\left(n_{io} - n_{e1} \, e^{i \, (\boldsymbol{k} \cdot \boldsymbol{r} - \omega \, t)}\right) \quad \text{and then,} \quad i \, (\boldsymbol{k} \cdot \boldsymbol{E}_1) = -\frac{e}{\epsilon_o} \, n_{e1} \tag{3.46}$$

With this linearized equation for the electric field fluctuation amplitude $\boldsymbol{E}_1$ we first explore the limit without thermal fluctuations ($T_e = 0$) setting $\nabla_{\mathbf{r}} \, p_e \sim 0$ in the momentum transport equation (3.45b). Using equation (3.46) and following the steps of section 3.3.1 we obtain from equation (3.45) the linearized equations,

$$\begin{aligned} -\,\omega \, n_{e1} + (\boldsymbol{k} \cdot \boldsymbol{u}_{e1}) \, n_o &= 0 \\ -\,i \, m_e \, \omega \, \boldsymbol{u}_{e1} + e \, \boldsymbol{E}_1 &= 0 \\ i \, (\boldsymbol{k} \cdot \boldsymbol{E}_1) + \frac{e}{\epsilon_o} \, n_{e1} &= 0 \end{aligned} \tag{3.47}$$

with the scalar product with $\boldsymbol{k}$ in the second equation and eliminating the perturbation amplitudes n_{e1}, $(\boldsymbol{u}_{e1} \cdot \boldsymbol{k})$ and $\boldsymbol{E}_1$ we find,

$$\omega = \left(\frac{e^2 \, n_o}{m_e \, \epsilon_o} \right)^{1/2}$$

We recover the electron plasma frequency introduced in section 4.5.2 of volume 1 as the faster response of electrons to external perturbations and its dispersion relation $\omega(k)$ is independent of the wave number. This means that the disturbance does not propagate in space, the electron low amplitude oscillations are not coupled so far collisions are neglected, as in the derivation of electron plasma frequency ω_{pe} in section 4.5.2 of volume 1.

The next step is to consider the electron temperature $T_e > 0$ using again the *adiabatic approximation* of the warm plasma model. For the pressure gradient $\nabla_{\mathbf{r}}\, p_e$ in equation (3.45b) we can write[12],

$$p_e = C\, n_e^{\gamma}, \qquad \frac{\nabla_{\mathbf{r}}\, p_e}{p_e} = \gamma\, \frac{\nabla_{\mathbf{r}}\, n_e}{n_e} \quad \text{and,} \quad \nabla_{\mathbf{r}}\, p_e = \gamma\, (k_B\, T_e)\, \nabla_{\mathbf{r}}\, n_e \tag{3.48}$$

When the electron pressure fluctuations are *isothermal* we have $\gamma = 1$, whereas $\gamma > 1$ for *adiabatic* transformations[13]. This last equation can be linearized as,

$$\nabla_{\mathbf{r}}\, p_e \sim \gamma\, (i\, \boldsymbol{k}\, n_1)\, e^{i\,(\boldsymbol{k}\cdot\boldsymbol{r} - \omega\, t)}$$

With this additional term in equation (3.45b) the linearized equations are now,

$$\begin{aligned} -\,\omega\, n_{e1} + (\boldsymbol{k}\cdot\boldsymbol{u}_{e1})\, n_o &= 0 \\ -\, i\, m_e\, n_o \omega\, \boldsymbol{u}_{e1} + e\, n_o\, \boldsymbol{E}_1 + \gamma\, (k_B T_e)\, n_1\, (i\, \boldsymbol{k}) &= 0 \\ i\, (\boldsymbol{k}\cdot\boldsymbol{E}_1) + \frac{e}{\epsilon_o}\, n_{e1} &= 0 \end{aligned} \tag{3.49}$$

Again, the elimination of perturbation amplitudes n_{e1}, $(\boldsymbol{u}_{e1}\cdot\boldsymbol{k})$ and $\boldsymbol{E}_1$ gives,

$$\omega^2 = \frac{e^2\, n_o}{m_e\, \epsilon_o} + \frac{\gamma\, k_B\, T_e}{m_e}\, k^2$$

and introducing the electron thermal velocity $v_{th} = \sqrt{2\, k_B\, T_e/m_e}$ we finally obtain,

$$\omega^2 = \omega_{pe}^2 + \frac{\gamma}{2}\, v_{th}^2\, k^2 \tag{3.50}$$

The phase velocity is $v_f = \omega/k$ and the group velocity v_g can be derived as,

$$2\, \omega\, d\omega = \frac{\gamma}{2}\, v_{th}^2\, (2\, k\, dk) \quad \text{then,} \quad v_g = \frac{\gamma}{2}\, v_{th}^2 \frac{k}{\omega} = \frac{\gamma}{2}\, \frac{v_{th}^2}{v_f}$$

Figure 3.2 shows schematically the relation between the dispersion relation $\omega(k)$ for electron plasma waves and the group and phase velocities. The wavenumber k_s gives the frequency $\omega_s = \omega(k_s)$ in equation (3.50) and $v_f = \omega_s/k_s$ is the slope of the dotted line. The group velocity is different from v_f and $v_g = d\omega/dk$ is the slope of the

[12] Since the ideal gas molecules have a Maxwell–Boltzmann energy distribution. Then, the equation of state $p_e = n_e\, k_B\, T_e$ is consistent with the assumption of a local Maxwellian electron population.
[13] See box 3.3, for the ideal gas law $p_e = k_B T_e\, n_e$ with T_e constant recovered taking $\gamma = 1$ and the adiabatic transformation $p_e = C\, n_e^{\gamma}$ corresponds to $\gamma > 0$ with C constant.

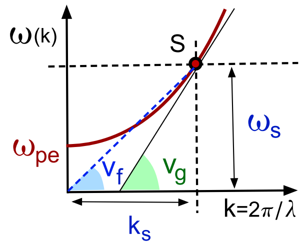

Figure 3.2. Dispersion relation for electron plasma waves.

solid line tangent to the point S. Both, v_g and v_f change as the point S in figure 3.2 moves along the curve (3.50) as $k = 2\pi/\lambda$ increases up to its maximum value for $\lambda \simeq \lambda_D$ corresponding to the minimum plasma scale length for perturbation amplitudes.

Introducing in the dispersion relation (3.50) the electron plasma frequency ω_{pe} we have,

$$\frac{\omega^2}{\omega_{pe}^2} = 1 + \frac{\gamma}{2}\left(\frac{2\,k_B T_e}{m_e}\right) \times \left(\frac{m_e\,\epsilon_o}{e^2\,n_o}\right) k^2 = 1 + \gamma\,\lambda_D^2\,k^2 = 1 + \gamma\,\frac{\lambda_D^2}{\lambda^2}$$

Then, for large perturbation amplitudes λ the frequency of electrons plasma waves is $\omega \simeq \omega_{pe}$ and this oscillation frequency increases as the length scale λ of fluctuations decreases up to a $\omega \simeq \omega_{pe}\sqrt{1+\gamma}$ maximum value for the plasma scale length.

3.4 The Langmuir sheath

Plasmas are limited by solid dielectric or metallic surfaces in nature and in the laboratory. Exposed surfaces of dielectric materials can accumulate electric charge and metallic walls biased at a fixed potential can drain and/or repel ions and electrons from the plasma. Alternatively, charged particles can be also collected by exposed metallic conductors connected to an external electric circuit.

When the electric potential of an exposed metallic surface differs from the distant plasma potential it is adjusted along a characteristic scale length. The steep change in the electric potential profile attached to the wall is called *Langmuir sheath* or *electrostatic plasma sheath*.

The plasma quasineutrality is not accomplished along the plasma sheath and its length scale is crucial for current collection processes from a plasma since the ions (or electrons) are accelerated or repelled by the intense and localized electric field. As figure 3.3 shows schematically, electrons and ions are accelerated or repelled by the intense electric field at the plasma boundaries. The calculation of its spatial potential profile is not trivial, even under the simplest conditions as shows the following one-dimensional model.

The metallic surface of figure 3.3 is negatively biased ($\varphi_w < \varphi_p$) with respect to an unmagnetized and PLE plasma ($T_e \gg T_i$). The ions move along the plasma potential profile $\varphi_p > \varphi(x) > \varphi_w$ and are collected by the wall placed at distance x_w from the

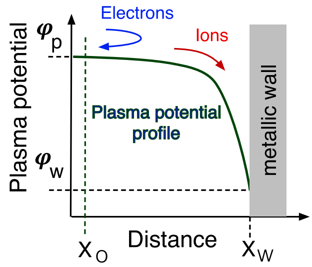

Figure 3.3. Plasma potential profile between the metallic wall and the undisturbed plasma. An intense electric field develops attached to the metallic wall.

unperturbed quasineutral plasma ($n_{eo} \cong n_{io}$) located at $x \leqslant x_o$ where $\varphi_p = \varphi(x)$ is considered uniform.

In low pressure plasmas the mean free path for short-range collisions with neutrals λ_c is usually much longer than the Debye length $\lambda_c \gg \lambda_D$ then, the motion of charges in the region $x_o \leqslant x \leqslant x_w$ of figure 3.3 is collisionless.

We can make use of equations (3.35) in one dimension for these *cold* ions and in this case we can drop the friction term $\boldsymbol{R}_i$ and also $(S_i - L_i) = 0$. In the stationary state we simply have,

$$\frac{\partial}{\partial x}(n_i\, u_i) = 0 \quad \text{and,} \quad m_i\, n_i\, u_i\, \frac{\partial u_i}{\partial x} = +e\, n_i\, E_x$$

where $E_x = -\partial\varphi/\partial x$ and $\varphi(x)$ is the electric potential.

The initial ion speed at x_o is u_{io} and we can write $n_{io}\, u_{io} = n_i(x)\, u_x(x)$ for the ion flux. The second equation can be integrated,

$$\frac{1}{2}m_i u_i^2(x) + e\,(\varphi(x) - \varphi_p) = \frac{1}{2}m_i u_{io}^2$$

so ion energy is preserved in the ion motion. The ion number density can be expressed as,

$$n_i(x) = n_{io}\, u_{io}\left(u_{io}^2 - \frac{2e\,(\varphi - \varphi_p)}{m_i}\right)^{-1/2} \tag{3.51}$$

Since we are interested in the slow time scale of the ion motion we can consider the fast electrons in thermal equilibrium as Maxwellian electron population as in box 3.2.

$$n_e(\varphi) = n_{eo}\exp[-e\,(\varphi - \varphi_p)/k_B T_e] \tag{3.52}$$

The plasma potential profile $\varphi(x)$ can be calculated using the Poisson equation as,

$$\epsilon_o\frac{d^2\varphi}{dx^2} = e\, n_{eo}\left[e^{(e(\varphi-\varphi_p)/k_B T_e)} - u_{io}\left(u_{io}^2 - \frac{2e\,(\varphi - \varphi_p)}{m_i}\right)^{-1/2}\right]$$

The physical magnitudes can be scaled using the ion acoustic speed $c_{ia} = \sqrt{k_B T_e / m_i}$ and,

$$\phi = -e(\varphi - \varphi_p)/K_B T_e > 0 \quad N_e = n_e/n_{eo} \quad N_i = n_i/n_{eo} \quad and, \quad V = u/c_{ia}$$

additionally $2e\varphi/m_i u_{io}^2 = 2\phi/(u_{io}/c_{ia})^2$ and $V_o = u_{io}/c_{ia}$, we obtain the following equation for the dimensionless potential,

$$\frac{d^2\phi}{dx^2} = -\frac{1}{\lambda_D^2}\left[e^{-\phi} - \left(1 + \frac{2\phi}{V_o^2}\right)^{-1/2}\right] \tag{3.53}$$

The distances x and λ_D are intentionally left explicit to show that $s = x/\lambda_D$ defines the length scale of the plasma sheath. The solution $\phi(x)$ of this non-linear differential equation gives the plasma potential profile of figure 3.3 in one dimension. However, no analytical solutions of equation (3.53) have been obtained for $\phi(x)$, however, after some transformations we will find an approximate solution.

3.4.1 Bohm criterion

We introduce $s = x/\lambda_D$ in equation (3.53) where $\mathcal{E} = -d\phi/ds$ represents the scaled electric field, which is normalized to the plasma sheath length scale. Multiplying by $d\phi/ds$ both sides of above equation we have,

$$\int_{so}^{s} \frac{d^2\phi}{ds^2}\frac{d\phi}{ds}ds = \left[\int_{so}^{s}\left(1 + \frac{2\phi}{V_o^2}\right)^{-1/2}\frac{d\phi}{ds}ds - \int_{so}^{s} \exp(-\phi)\frac{d\phi}{ds}ds\right]$$

where $s_o \leqslant s \leqslant s_w$ is the dimensionless coordinate of a point located between s_o and the wall $s_w = x_w/\lambda_D$ in figure 3.3. The integration gives,

$$\frac{1}{2}\left[\mathcal{E}^2(s) - \mathcal{E}^2(s_o)\right] = \frac{1}{2}\left(\frac{d\phi}{ds}\right)^2\Bigg]_{s_o}^{s} = V_o^2\sqrt{1 + \frac{2\phi}{V_o^2}}\,\Bigg]_{s_o}^{s} + \exp(-\phi)]_{s_o}^{s}$$

The scaled electric field $\mathcal{E}(s) = -d\phi/ds$ is proportional to the slope of the plasma potential profile in figure 3.3 and $\mathcal{E}(s) \gg \mathcal{E}(s_o)$. Then, the term that gives the lower limit of this integral can be neglected since it is small compared with the upper limit term. For points $s > s_o$ within the plasma sheath we have,

$$\frac{1}{2}\mathcal{E}^2(s) = \frac{1}{2}\left(\frac{d\phi}{ds}\right)^2 = V_o^2\left[\sqrt{1 + \frac{2\phi}{V_o^2}} - 1\right] + (\exp(-\phi) - 1) > 0$$

which is always positive. The non-linear differential equation (3.53) for $\phi(x)$ is transformed into a non-linear algebraic equation that can only be solved numerically if we make some approximations. For *ideal plasmas* where $k_B T_e \gg e(\varphi - \varphi_p)$ (or $\phi \ll 1$) the right term can expanded in powers as,

$$\frac{1}{2}\left(\frac{d\phi}{ds}\right)^2 = V_o^2\left[\left(1 + \frac{\phi}{V_o^2} - \frac{1}{2}\frac{\phi^2}{V_o^4} + \cdots\right) - 1\right]$$
$$\times\left[(1 - \phi + \frac{1}{2}\phi^2 + \cdots) - 1\right] > 0$$

and holding the two first powers we have,

$$V_o^2\left[\frac{\phi}{V_o^2} - \frac{1}{2}\frac{\phi^2}{V_o^4}\right] - \phi + \frac{1}{2}\phi^2 > 0$$

and hence,

$$\left[-\frac{1}{V_o^2} + 1\right]\frac{1}{2}\phi^2 > 0$$

For the last equation to be positive the normalized ion speed has to be $V_o > 1$, that is,

$$V_o > 1, \quad \text{or,} \quad u_{io} > c_{ia} \tag{3.54}$$

This inequality is called *Bohm sheath criterion*, and shows that the speed of ions at point x_o must be *higher* that the ions' acoustic velocity.

We have not derived an *explicit* expression for $\varphi(x)$ but to fulfill this condition an electric field must accelerate the ions to supersonic velocities ($u_{io} > c_{ia}$) *before* the ions enter into the sheath region. Therefore, the plasma potential profile of figure 3.3 is not monotonic and has at least two parts; a *presheath* region before the electron-free zone where the large potential drop of the *plasma sheath* takes place.

Since the electric field is null at s_o, the *presheath* region is located between s_o and the undetermined point s where the condition (3.54) is fulfilled before the large voltage drop associated with the *plasma sheath*. The plasma potential drop along the *presheath* is smooth as,

$$\frac{1}{2}\, m_i\, c_{is}^2 = k_B\, T_e \sim e\, \Delta\varphi \quad \text{then,} \quad \Delta\varphi \sim \frac{k_B\, T_e}{e}$$

is in the order of the electron thermal energy.

The physical interpretation of the Bohm criterion $u_{io} > c_{ia}$ can be understood with the help of figure 3.4 where the logarithm of electron (3.52) and ion (3.51) plasma densities are represented against the dimensionless potential. The Maxwellian electron number density $n_e = n_{eo}\exp(-\phi)$ is a straight line with slope (-1) and the ion density from equation (3.51),

$$N_i = \left(1 + \frac{2\,\phi}{V_o^2}\right)^{-1/2} \quad \text{and,} \quad \ln(n_i) = \ln(n_{io}) - \frac{1}{2}\ln\left(1 + \frac{2\,\phi}{V_o^2}\right)$$

decreases more slowly with the dimensionless potential. Both start at $\phi = 0$ ($\varphi = \varphi_p$) where $n_{io} = n_{eo}$ and the charge density $\rho = e\,(n_i - n_e) > 0$ must be positive, so n_i

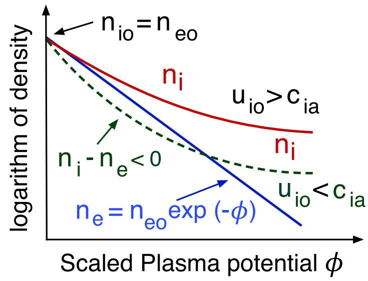

Figure 3.4. The $n_i(\phi)$ and electron $n_e(\phi)$ densities as a function of ϕ.

must always lie over the line corresponding to the electron density. For this to happen the slope of $n_i(\phi)$ at $\phi = 0$,

$$\frac{d}{d\phi}\left[\frac{1}{2}\left(1 + \frac{2\,\phi}{V_o^2}\right)\right] = -\frac{1}{V_o^2} < (-1)$$

needs to be below (−1) and $v_o > 1$, otherwise the electric charge density reverts its sign as in the dotted line of figure 3.4.

So far we have considered a *collisionless* plasma sheath where $\lambda_c \gg \lambda_D$, which governs the thickness of the plasma sheath. When short-range collisions are important the Bohm criterion needs to be generalized and this subject is beyond the scope of the present work.

3.4.2 Child–Langmuir law

The values for $\phi(s)$ are large at points s close to the wall and far from the presheath where the electric potential profile of figure 3.4 becomes steeper. In the Poisson equation (3.53) we can neglect the contribution of electrons ~exp(−ϕ) in this zone where $2\phi/V_o^2 \gg 1$ and,

$$\frac{d^2\phi}{ds^2} = \left(1 + \frac{2\phi^2}{V_o^2}\right)^{-1/2} \cong \frac{V_o}{\sqrt{2\phi}}$$

Multiplying by $d\phi/ds$ on both sides we can integrate between the point s_i where the electron density is neglected and $s \leqslant s_w$ inside the sheath and close to the wall at $s_w = x_w/\lambda_D$,

$$\int_{s_i}^{s} \frac{d^2\phi}{ds^2}\,\frac{d\phi}{ds}\,ds = \int_{s_i}^{s} \frac{V_o}{\sqrt{2\phi}}\,\frac{d\phi}{ds}\,ds \quad \text{then,} \quad \frac{1}{2}\left(\frac{d\phi}{ds}\right)^2\Bigg]_{s_i}^{s} = \sqrt{2}\;V_o\,\sqrt{\phi}\,\Bigg]_{s_i}^{s}$$

Using the dimensionless electric field $\mathcal{E} = -d\phi/ds$ we have,

$$\frac{1}{2}\left[\mathcal{E}^2(s) - \mathcal{E}^2(s_i)\right] = \sqrt{2}\;V_o\left[\sqrt{\phi(s)} - \sqrt{\phi_i}\right]$$

We neglect $\mathcal{E}^2(s_i)$ compared to $\mathcal{E}^2(s)$ and also $\phi(s) \gg \phi(s_i)$ and the resulting differential equation,

$$\frac{d\phi}{ds} = 2^{3/4}\sqrt{V_o}\,\phi^{1/4}$$

can be integrated for points s close to the wall with electric potential $\phi_w = \phi(s_w)$,

$$\frac{4}{3}\left[\phi_w^{3/4} - \phi^{3/4}\right] = 2^{1/2}\sqrt{V_o}\,(s_w - s)$$

The solution provides an expression for $\phi(s)$ which depends on the ion velocity $V_o = u_{io}/c_{ia}$ and the bias potential of the wall. Neglecting $\phi(s) \ll \phi_w$ we obtain a relation between V_o and ϕ_w as,

$$V_o = \frac{4\sqrt{2}}{9}\frac{\phi_w^{3/2}}{(s_w - s)^2} \quad \text{and substituting,}$$

$$u_{io} = \frac{4\sqrt{2}}{9}\left(\frac{\epsilon_o K_B T_e}{e^2 n_{eo}}\right)\frac{1}{d^2}\sqrt{\frac{K_B T_e}{m_i}}\left(\frac{e\,\varphi_w}{K_B T_e}\right)^{3/2}$$

Since the ion current density is conserved in the sheath we can write $J_i = n_i u_i = n_{io} u_{io}$ and finally we obtain,

$$J_i = \frac{4}{9}\frac{\epsilon_o}{e^2}\sqrt{\frac{2e}{m_i}}\frac{\varphi_w^{3/2}}{d^2} \tag{3.55}$$

which is the *Child–Langmuir law* for a plane diode.

The ion current density J_i from the plasma scales with the 3/2 power of the electric potential φ_w of the wall (or electrode) located at a distance $d = x_w - x$ where x is the point where we consider the potential and the electric field as negligible. This Child–Langmuir law has been accurately verified and also applies to the current of electrons or ions transported between two electrodes separated at distance d.

Similar arguments apply when ions are repelled by the conductive wall of figure 3.3 and replacing m_i by the electron mass m_e in equation (3.55) is also valid for the electron current density.

Turning back to figure 3.3 the potential profile close to the conductive wall can be divided into three zones. Since electrons are repelled, the zone closest to the wall is electron-free and its thickness is given by equation (3.55). Next comes a zone of thickness λ_D a where n_e is appreciable and finally the *presheath* where ions are accelerated up to the Bohm speed (3.54). Its extension is much larger than the *sheath* as the potential drop for ion acceleration along the *presheath* $\Delta\varphi k_B T_e/e$ is smooth.

3.4.3 Space-charge neutralization

The *space charge* refers to the excess of positive or negative electric charge concentrated on a region of space, which produces a localized electric field that rejects charges of the same sign. The maximum current density (positive or negative) that can be transported in vacuum in the steady given by equation (3.55) is said to be *space-charge* limited.

The above Child–Langmuir law gives the maximum ion (or electron) current that can be transported in vacuum under the electric potential drop φ. This is a limiting factor for charged particle sources and specifically for plasma thrusters, since the maximum ion flow that delivers thrust can be bounded by charge space effects

The term *neutralization* refers to mixing positively and negatively charged particles to mitigate or to eliminate these electric fields produced by charged beams. The stream of particles can be *space-charge* neutralized, *current* neutralized or both. In the first case, the mix of positive and negative charges decreases the average beam electric field and the current of opposite charged particle lowers the beam magnetic field in the second case. The result of the neutralization process is a plasma stream of electrons and ions where the quasineutrality condition $n_e \simeq n_i$ holds. Therefore, equation (3.55) is no longer valid as it only accounts for one charged species [7].

As we shall see in chapter 6, electron-emitting cathodes are used in the *space charge neutralization* of ion beams to form plasma thrusters, as the magnetic fields produced are low. This neutralization process creates a quasineutral plasma flow where the ion momentum losses are negligible due to the smallness of the electron mass.

3.5 Commentaries and further reading

The hydrodynamic plasma models accurately describe a large number of physical phenomena despite the gross simplifications made. The derivation of the transport fluid equations essentially comes from reference [5] and an introductory approach is in [2]. An updated and rigorous formulation is found in reference [3] and in the classical book [4] where the calculation of friction force is in chapter 6. The MHD equations are of interest in the study of plasma flows at planetary and galactic length scales and its formulation can be found in the books [2] (section 4.13), [8] (chapter 9) and [9] (chapter 6).

The asymptotic closure of transport equations for neutral gases is discussed in the book [4]. For magnetized plasma see the classical review [5] of Braginskii, and more recently sections 4.10 and 4.11 of reference [2]. The book [10] is a comprehensive and up-to-date formulation of plasma waves theory and experiments.

The study of *plasma sheaths* is of interest in a number of fields. For example, the conductive walls of spacecraft or their exposed metal surfaces can drain electrical charge from the ambient space plasma. In this process called *spacecraft charging* the vehicle or some of the elements can acquire high voltages that produce unwanted currents and/or trigger electric discharges. Reference [11] is an overview of space charging phenomena and [12] a comprehensive book on this subject. Finally, reference [13] is an excellent review on plasma sheaths in the context of space plasmas.

Initially used for the design of vacuum valves, the applications of the Child–Langmuir equation for a plane diode (3.55) have found a countless number of applications. The classical one-dimensional model for a plane sheath is tractable, but the extension of the Child–Langmuir law to more dimensions is difficult, as shown in reference [14]. Additionally, the generalization of Bohm sheath criterion to account for elastic or ionizing collisions in the plasma sheath region is still a subject of current studies.

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