Máster Universitario en Ingeniería Aeronáutica

The Space Environment

Hydrodynamic description of plasmas. The continuity equation

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Continuum models

$$K_n = \frac{\lambda_c}{L_s} \ll 1$$

- Ions, electrons and neutral atoms are considered as three mutually interpenetrating fluids whose motions are coupled by short-range collisions and long-range electromagnetic interactions.
- The physical magnitudes are ($\alpha = e, i, a$) are macroscopic: $\boldsymbol{u}_\alpha(\boldsymbol{r}, t)$, $n_\alpha(\boldsymbol{r}, t)$, $T_\alpha(\boldsymbol{r}, t)$.
- Electric field shielding requires $N_\alpha = n_\alpha(r,t)\,\lambda_D^3$ of countless charges within the Debye length scale.

Length scale $\ell \ll L_s$ small compared with the dimensions L_s of the system, but longer than λ_{De}

- $L_s \gg \ell \gtrsim \lambda_D$
- $L_s \gg \ell \gg \lambda_c$
- $N_\alpha = n_\alpha(r,t)\,\ell^3$ is huge

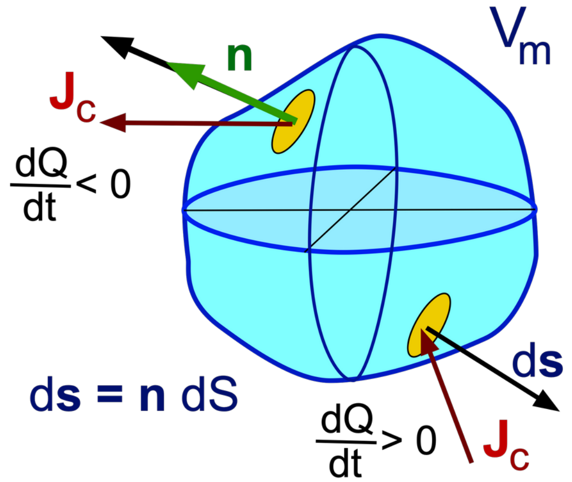
In these conditions, *collisions relax small random fluctuations* of energy $\delta E_\alpha \sim \delta T_\alpha \ll T_\alpha$ and densities $\delta n_\alpha \ll n_\alpha$ over a time scale $\tau \ll T_s \sim L_s/|\boldsymbol{u}_\alpha|$ much smaller than the macroscopic time scale T_s associated to time evolution of the system.

The *continuity* or *mass conservation* equation:

$$\frac{\partial n_\alpha}{\partial t} + \nabla\cdot(n_\alpha\,\boldsymbol{u}_\alpha) = S_\alpha - L_\alpha$$

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla\cdot(\rho_\alpha\,\boldsymbol{u}_\alpha) = m_\alpha\,(S_\alpha - L_\alpha)$$

- Here S_α and L_α are the source and loss terms.
- The difference $(S_\alpha - L_\alpha)$ is the net rate at which particles of type α are added per unit volume.



The difference $(S_\alpha - L_\alpha) = 0$ implies mass/particle conservation (also for electric charge).

$$\frac{\partial}{\partial t}\left[\int_{V_m} \rho_\alpha\, dV\right] + \left(\int_{V_m} \nabla\cdot(\rho_\alpha\, u_\alpha)\, dV\right) = 0 \qquad \frac{\partial M_\alpha}{\partial t} = -\int_S \boldsymbol{J}_m\cdot d\boldsymbol{S}$$

$$\begin{cases} \boldsymbol{J}_m\cdot d\boldsymbol{S} < 0 \rightarrow \dot{M}_\alpha > 0 \\ \boldsymbol{J}_m\cdot d\boldsymbol{S} > 0 \rightarrow \dot{M}_\alpha < 0 \end{cases}$$

Generalization of collision frequency concept

- We consider a Total cross section: $\sigma_{ab} = \pi(r_a + r_b)^2$
- Target "a" particles are not at rest in the laboratory frame, thus $\boldsymbol{v}_a - \boldsymbol{v}_b$ is their relative velocity

$$\nu_{ab} = \sigma_{ab}\, n_a n_b\, |\boldsymbol{v}_a - \boldsymbol{v}_b| = \sigma_{ab} \times n_a \times \underbrace{(n_b|\boldsymbol{v}_a - \boldsymbol{v}_b|)}$$

Is the flux of projectile "b" particles in a frame where "a" targets are at rest.

$$\Gamma_b = n_b\, |\boldsymbol{v}_a - \boldsymbol{v}_b|$$

Thus, we can write $\nu_{ab} = \sigma_{ab} \times n_a \times \Gamma_b$ and $\nu_{ab} = \nu_{ba}$ since we can switch "a" by "b" in this equation.

- Generalization: Colliding species have a stationary velocity distribution and $\sigma_{ab}(|\boldsymbol{v}_a - \boldsymbol{v}_b|)$ is a function,

$$n_a \rightarrow dn_a = n_{ao}\, f_a(\boldsymbol{v}_a)\, d^3 v_a$$

$$n_b \rightarrow dn_b = n_{bo}\, f_b(\boldsymbol{v}_b)\, d^3 v_b$$

$$\nu_{ab} = \sigma_{ab} \times dn_a \times d\Gamma_b = \sigma_{ab} \times (dn_a) \times [(dn_b) \times |\boldsymbol{v}_a - \boldsymbol{v}_b|]$$

$$\nu_{ab} = \sigma_{ab} \times (n_{ao}\, f_a(\boldsymbol{v}_a)\, d^3 v_a) \times [n_{bo}\, f_b(\boldsymbol{v}_b) \times |\boldsymbol{v}_a - \boldsymbol{v}_b| d^3 v_b]$$

$$\langle \nu_{ab} \rangle = n_{ao}\, n_{bo} \iint \sigma_{ab}(|\boldsymbol{v}_a - \boldsymbol{v}_b|)\, f_a(\boldsymbol{v}_a)\, f_b(\boldsymbol{v}_b)\, |\boldsymbol{v}_a - \boldsymbol{v}_b|\, d^3 v_a d^3 v_b = n_{ao}\, n_{bo}\, K_{ab}$$

This average $\langle \nu_{ab} \rangle$ represents the number of "ab" collisions by time and volume units.

- The integral $K_{ab} = \langle \sigma_{ab} |\boldsymbol{v}_a - \boldsymbol{v}_b| \rangle$ is called the *reaction rate*.
- Elastic collisions: all velocities of particles $(\boldsymbol{v}_a, \boldsymbol{v}_b)$ are considered, and above integrals are extended to infinity.
- Inelastic collisions: A minimum velocity for the projectile particle is required.

Ionization frequency

- In a background of cold ions and neutrals and electrons where $T_e \gg T_i \sim T_a$ with relative velocity $\boldsymbol{u}_o$ in the laboratory frame
  - Target "a" particles (neutrals): $dn_a = n_{ao}\,\delta_e(\boldsymbol{v}_a - \boldsymbol{u}_o)\,d^3v_a$
  - Projectile "b" particles (electrons) $dn_e = n_{eo}\,f_e(\boldsymbol{v}_e)\,d^3v_e$

$$d\langle \nu_I \rangle = \sigma_I \times dn_a \times \Gamma_e = \underbrace{\sigma_I(|\boldsymbol{v}_a - \boldsymbol{v}_e|)}_{\text{Ionization Cross section}} \times \underbrace{[n_{ao}\,\delta_a(\boldsymbol{v}_a - \boldsymbol{u}_o)\,d^3v_a]}_{\text{Background of cold neutral gas atoms}} \times \underbrace{[n_{eo}\,f_e(\boldsymbol{v}_e)\,d^3v_e] \times |\boldsymbol{v}_a - \boldsymbol{v}_e|]}_{\Gamma_e = dn_e\,|\boldsymbol{v}_a - \boldsymbol{v}_e| \text{ Flux of electrons}}$$

- Electrons need of a *minimum velocity* $v_I = \sqrt{2E_I/m_e}$ to produce one ionization event.
- Neutral atoms can be ionized for all velocities $\boldsymbol{v}_a$ the integration of,

$$d\langle \nu_I \rangle = [n_{eo}\,f_e(\boldsymbol{v}_b) d^3v_e] \times \int_{-\infty}^{+\infty} \sigma_I(|\boldsymbol{v}_a - \boldsymbol{v}_e|) \times |\boldsymbol{v}_a - \boldsymbol{v}_e| \times [n_{ao}\,\delta_a(\boldsymbol{v}_a - \boldsymbol{u}_o)]\,d^3v_a$$

using the Dirac delta property $g(\boldsymbol{r}_o) = \int_{-\infty}^{+\infty} g(\boldsymbol{r})\,\delta(\boldsymbol{r} - \boldsymbol{r}_o)\,d^3r$ gives, $g(\boldsymbol{r}_o) \to n_{ao}\,\sigma_I(|\boldsymbol{u}_o - \boldsymbol{v}_e|) \times |\boldsymbol{u}_o - \boldsymbol{v}_b|$

$$\langle \nu_I \rangle = n_{ao}\,n_{eo} \int_{V_I}^{+\infty} \sigma_I(|\boldsymbol{u}_o - \boldsymbol{v}_e|) \times |\boldsymbol{u}_o - \boldsymbol{v}_e| \times [f_e(\boldsymbol{v}_e)]\,d^3v_e$$

For the neutral gas is at rest ($\boldsymbol{u}_o = 0$) $\langle \nu_I \rangle = n_{ao}\,n_{eo} \int_{V_I}^{+\infty} \sigma_I(|\boldsymbol{v}_e|)\ |\boldsymbol{v}_e|\,f_e(\boldsymbol{v}_e)\,d^3v_e$

Ionization in a fluid model

- The source term in the continuity equation is the average $\langle \nu_I \rangle = S_e$ representing the number of ionizing collisions by time and volume units.

$$\begin{cases} S_e = \langle \nu_I \rangle = n_{ao}\, n_{eo} \displaystyle\int_{V_I}^{+\infty} \sigma_I(|\boldsymbol{v}_e|)\ |\boldsymbol{v}_e|\, f_e(\boldsymbol{v}_e)\, d^3v_e \\ S_e = \langle \nu_I \rangle = K_I\, n_{ao}\, n_{eo} = Q\, n_{eo} \end{cases}$$

Approximations:

Assuming a piecewise ionization cross section:

$$\sigma_I(v_e) = \begin{cases} \sigma_o & \text{if } |v_e| \geq V_I \\ 0 & \text{if } |v_e| < V_I \end{cases} \qquad S_e \sim Q \times n_{eo} = n_{eo} n_{ao}\, \sigma_o \int_{V_I}^{+\infty} |\boldsymbol{v}_e|\, f_e(\boldsymbol{v}_e)\, d^3v_e$$

Sometimes it is also assumed:

$$\int_{V_I}^{+\infty} |\boldsymbol{v}_e|\, f_e(\boldsymbol{v}_e)\, d^3v_e \sim V_{th,e} = \sqrt{\frac{8\, k_B T_e}{\pi m_e}} \qquad S_e \sim n_{eo} n_{ao}\, \sigma_o\, V_{th,e}$$

When the plasma is weakly ionized ($n_{ao} \gg n_e$) and not flowing ($\boldsymbol{u}_e = \boldsymbol{u}_i = 0$)

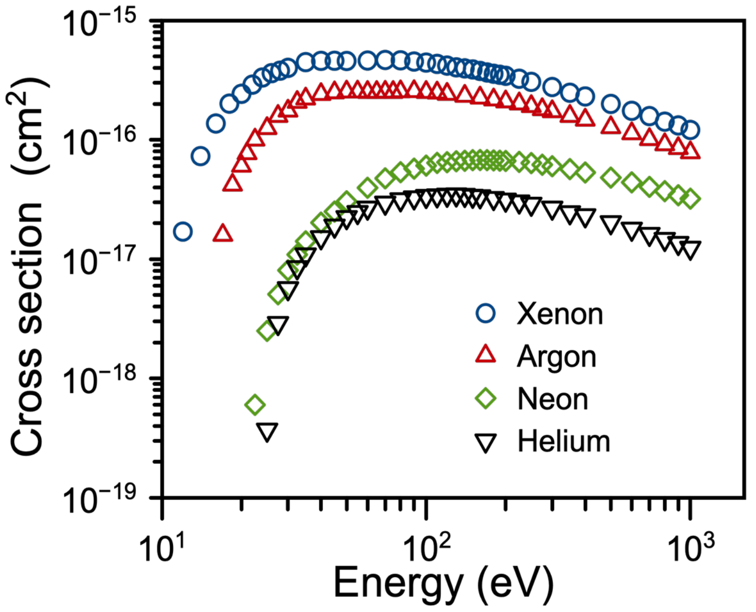
$$\left.\begin{aligned} \frac{\partial n_e}{\partial t} + \nabla \cdot (n_e\, \boldsymbol{u}_e) = K_I\, n_{ao}\, n_e \rightarrow \frac{\partial n_e}{\partial t} = Q\, n_e \\ \frac{\partial n_i}{\partial t} + \nabla \cdot (n_i\, \boldsymbol{u}_i) = K_I\, n_{ao}\, n_e \rightarrow \frac{\partial n_i}{\partial t} = Q\, n_e \end{aligned}\right\} \quad n_e(t) = e^{Qt}$$

Charge production is exponential!

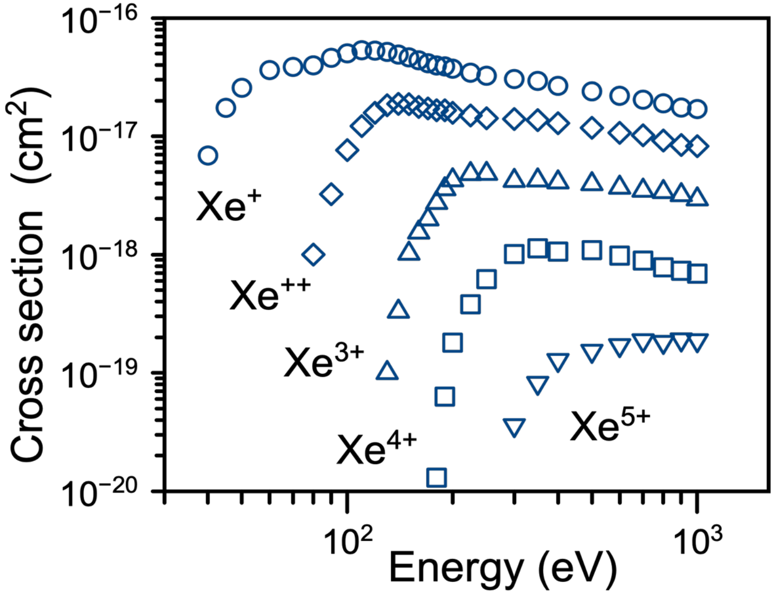
Electron impact ionization

- Electrons are produced by electron impact ionization of neutrals: $e^- + A \rightarrow A^+ + 2\,e^-$
- The ionizing electron needs energy over the ionization energy threshold E_I, (typically of tens of eV $(15.76\text{ eV for Argon})$
- The typical cross sections for the first ionization level of noble gases are $\sigma_I \sim (5-50)\times 10^{-16}\text{ cm}^2$

Cross sections for noble gases



Cross sections for different ionization levels of Xe



Estimation:

- The mean free path for ionizing collisions is, $$\lambda_I = \frac{1}{\sigma_I\, n_a}$$
- The number of ionizing collisions for one electron with velocity $v_e > \sqrt{2E_I/m_e}$ is, $$\nu_I = \frac{v_e}{\lambda_I} = \frac{1}{\tau_c} = \sigma_I\, n_a v_e$$

For n_e electrons

$$S_e = \nu_I \times n_e = \sigma_I\, n_a\, n_e\, v_e$$

Represents the number of ionization events per volume by time unit

Photoionization

$$A + h\,\nu \longrightarrow A^{+} + e$$

Interaction of light with an atom results in an ionization event.

The photon has enough energy to *strip* an electron from the atom.

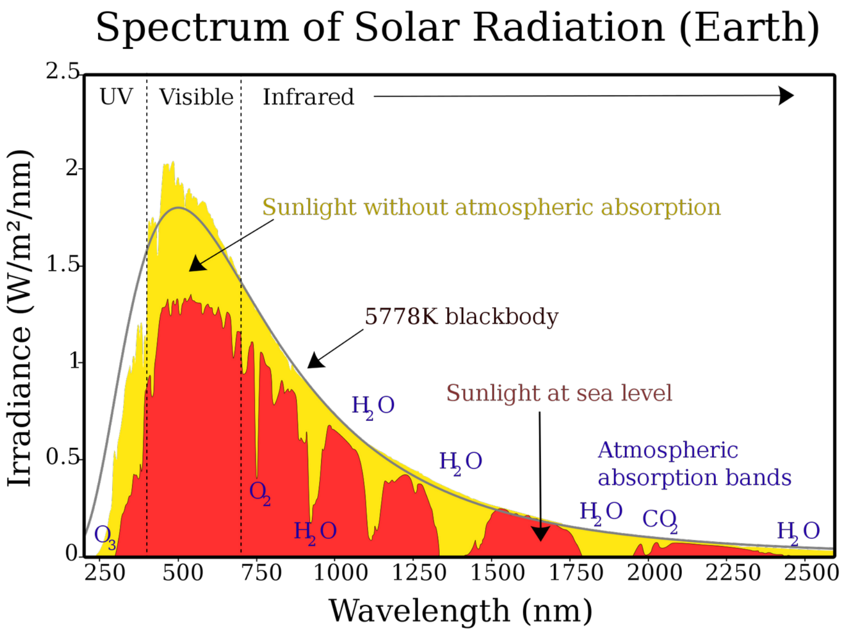
Usually, the energy of the photon needs to be above the first ionization energy.

Extremely important in LEO

Solar radiation and cosmic rays

$$E_{ph} = \frac{hc}{\lambda_{ph}}$$

$$10\,\text{eV} \rightarrow 123\,\text{nm};\;\; 20\,\text{eV} \rightarrow 62\,\text{nm}$$



Three-body recombination (electron-ion)

- One electron *slows down* another causing it to recombine: $2\,e^- + A^+ \rightarrow A^* + e^-$
- The typical cross sections are much lower than for ionizing collisions,

$$K_R \sim 10^{-7}\,\text{cm}^3\ \text{s}^{-1} \ll K_I$$

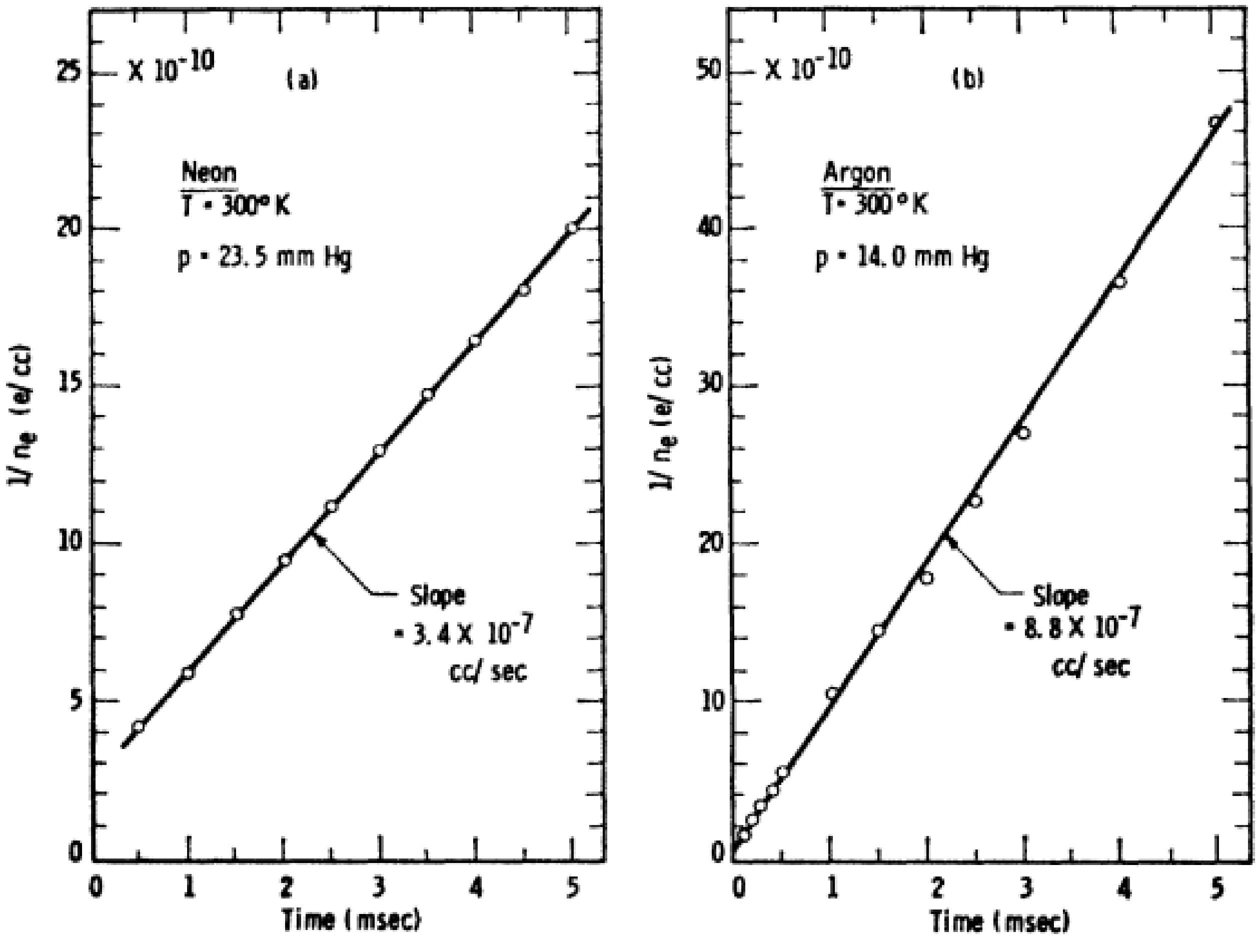
$$L_e = K_R\, n_e\, n_i$$

- When the plasma is weakly ionized ($n_{ao} \gg n_e$) and not flowing ($\boldsymbol{u}_e = \boldsymbol{u}_i = 0$) in the fluid model assuming a quasineutral plasma $n_e = n_i = n(t)$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e\, \boldsymbol{u}_e) = -K_R\, n_e n_i \quad \rightarrow \quad \frac{\partial n}{\partial t} = -K_R n^2$$

and with the initial plasma density, $n(0) = n_o$

$$\frac{dn}{dt} = -K_R\, n^2 \qquad\qquad \frac{1}{n(t)} = \frac{1}{n_o} + K_R\, t$$



Radiative and Dielectronic recombination

$$\mathrm{e} + \mathrm{A}^{q+1} \longrightarrow \mathrm{A}^{q} + h\,\nu$$ Radiative recombination

$$\mathrm{e} + \mathrm{A}^{q+1} \longrightarrow \mathrm{A}^{\mathrm{q}**} \longrightarrow \mathrm{A}^{q} + h\,\nu$$ Dielectronic recombination

Both processes share the emission of a photon (of frequency ν) after the recombination.
The frequency of emission depends on the exact transition.
In dielectronic recombination the ion goes through an intermediate excited state.
The de-excitation process is the one emitting the light.
Where does the emitted radiation go?

Saha equation for a plasma in equilibrium

Developed by Meghnad Saha in 1920 for stellar plasmas.

CONDITION: Ionization comes from collisions of thermal atoms.

$$\frac{n_{i+1} n_e}{n_i} = \frac{2}{\lambda_{th}^3} \frac{g_{i+1}}{g_i} \exp\left[-\frac{\epsilon_{i+1} - \epsilon_i}{kT}\right]$$

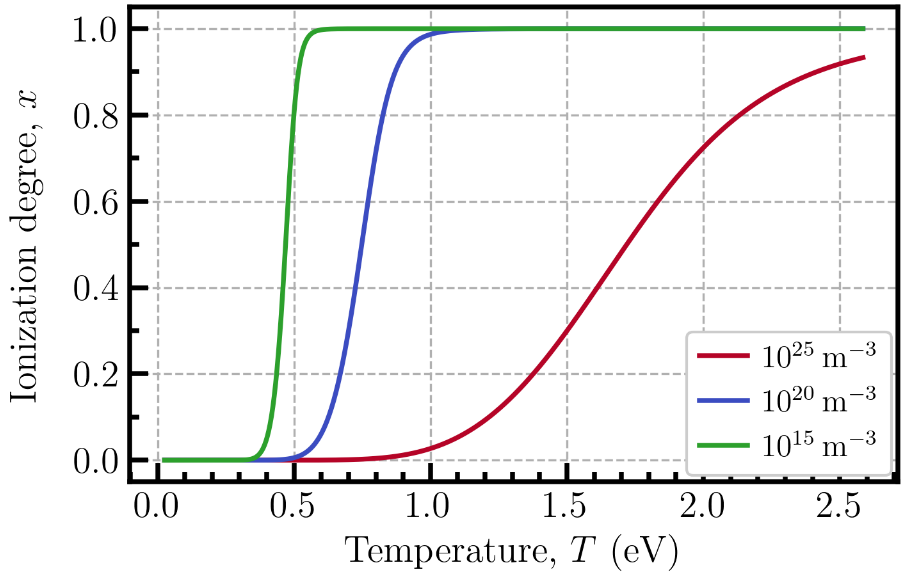
- n_i, n_e are the ion/electron density
- g_i s the degree of degeneracy of the i state
- ϵ_i is the ionization energy
- kT is the temperature (in energy units)
- λ_{th} is the thermal de Broglie wavelength

For a single species plasma, n_0, n_1, n_e (H, H$^+$, e):

$$\lambda_{th} = \frac{h}{\sqrt{2\pi m_e kT}}$$

$$n_1 = n_e;\ n = n_0 + n_1;\ g_1 = g_0;\ \frac{n_e^2}{n - n_e} = \frac{2}{\lambda_{th}^3} \exp\left[-\frac{\epsilon}{kT}\right]$$

$$x = \frac{n_1}{n} = \frac{n_e}{n};\ \frac{x^2}{1-x} = \frac{2}{n\lambda_{th}^3} \exp\left[-\frac{\epsilon}{kT}\right]$$



$\epsilon = 13.6\,\text{eV}$ **How is it possible?**

Why x increases at lower n ?