Generalization of collision frequency concept

- We consider a Total cross section: $\sigma_{ab} = \pi(r_a + r_b)^2$
- Target "a" particles are not at rest in the laboratory frame, thus $\boldsymbol{v}_a - \boldsymbol{v}_b$ is their relative velocity

$$\nu_{ab} = \sigma_{ab}\, n_a n_b\, |\boldsymbol{v}_a - \boldsymbol{v}_b| = \sigma_{ab} \times n_a \times \underbrace{(n_b|\boldsymbol{v}_a - \boldsymbol{v}_b|)}$$

Is the flux of projectile "b" particles in a frame where "a" targets are at rest. $\Gamma_b = n_b\,|\boldsymbol{v}_a - \boldsymbol{v}_b|$

Thus, we can write $\nu_{ab} = \sigma_{ab} \times n_a \times \Gamma_b$ and $\nu_{ab} = \nu_{ba}$ since we can switch "a" by "b" in this equation.

- Generalization: Colliding species have a stationary velocity distribution and $\sigma_{ab}(|\boldsymbol{v}_a - \boldsymbol{v}_b|)$ is a function,

$$n_a \to dn_a = n_{ao}\, f_a(\boldsymbol{v}_a)\, d^3v_a$$

$$n_b \to dn_b = n_{bo}\, f_b(\boldsymbol{v}_b)\, d^3v_b$$

$$\nu_{ab} = \sigma_{ab} \times dn_a \times d\Gamma_b = \sigma_{ab} \times (dn_a) \times [(dn_b) \times |\boldsymbol{v}_a - \boldsymbol{v}_b|]$$

$$\nu_{ab} = \sigma_{ab} \times (n_{ao}\, f_a(\boldsymbol{v}_a)\, d^3v_a) \times [n_{bo}\, f_b(\boldsymbol{v}_b) \times |\boldsymbol{v}_a - \boldsymbol{v}_b| d^3v_b]$$

$$\langle \nu_{ab} \rangle = n_{ao}\, n_{bo} \iint \sigma_{ab}(|\boldsymbol{v}_a - \boldsymbol{v}_b|)\, f_a(\boldsymbol{v}_a)\, f_b(\boldsymbol{v}_b)\, |\boldsymbol{v}_a - \boldsymbol{v}_b|\, d^3v_a d^3v_b = n_{ao}\, n_{bo}\, K_{ab}$$

This average $\langle \nu_{ab} \rangle$ represents the number of "ab" collisions by time and volume units.

- The integral $K_{ab} = \langle \sigma_{ab} |\boldsymbol{v}_a - \boldsymbol{v}_b| \rangle$ is called the *reaction rate*.
- Elastic collisions: all velocities of particles $(\boldsymbol{v}_a, \boldsymbol{v}_b)$ are considered, and above integrals are extended to infinity.
- Inelastic collisions: A minimum velocity for the projectile particle is required.

Ionization frequency

- In a background of cold ions and neutrals and electrons where $T_e \gg T_i \sim T_a$ with relative velocity $\boldsymbol{u}_o$ in the laboratory frame
  - Target "a" particles (neutrals): $dn_a = n_{ao}\,\delta_e(\boldsymbol{v}_a - \boldsymbol{u}_o)\,d^3v_a$
  - Projectile "b" particles (electrons) $dn_e = n_{eo}\,f_e(\boldsymbol{v}_e)\,d^3v_e$

$$d\langle \nu_I\rangle = \sigma_I \times dn_a \times \Gamma_e = \underbrace{\sigma_I(|\boldsymbol{v}_a - \boldsymbol{v}_e|)}_{\text{Ionization Cross section}} \times \underbrace{[n_{ao}\,\delta_a(\boldsymbol{v}_a - \boldsymbol{u}_o)\,d^3v_a]}_{\text{Background of cold neutral gas atoms}} \times \underbrace{[n_{eo}\,f_e(\boldsymbol{v}_e)\,d^3v_e] \times |\boldsymbol{v}_a - \boldsymbol{v}_e|]}_{\Gamma_e = dn_e\,|\boldsymbol{v}_a - \boldsymbol{v}_e| \text{ Flux of electrons}}$$

- Electrons need of a *minimum velocity* $v_I = \sqrt{2E_I/m_e}$ to produce one ionization event.
- Neutral atoms can be ionized for all velocities $\boldsymbol{v}_a$ the integration of,

$$d\langle \nu_I\rangle = [n_{eo}\,f_e(\boldsymbol{v}_b)d^3v_e] \times \int_{-\infty}^{+\infty} \sigma_I(|\boldsymbol{v}_a - \boldsymbol{v}_e|) \times |\boldsymbol{v}_a - \boldsymbol{v}_e| \times [n_{ao}\,\delta_a(\boldsymbol{v}_a - \boldsymbol{u}_o)]\,d^3v_a$$

using the Dirac delta property $g(\boldsymbol{r}_o) = \int_{-\infty}^{+\infty} g(\boldsymbol{r})\,\delta(\boldsymbol{r} - \boldsymbol{r}_o)\,d^3r$ gives, $g(\boldsymbol{r}_o) \rightarrow n_{ao}\,\sigma_I(|\boldsymbol{u}_o - \boldsymbol{v}_e|) \times |\boldsymbol{u}_o - \boldsymbol{v}_b|$

$$\langle \nu_I\rangle = n_{ao}\,n_{eo} \int_{V_I}^{+\infty} \sigma_I(|\boldsymbol{u}_o - \boldsymbol{v}_e|) \times |\boldsymbol{u}_o - \boldsymbol{v}_e| \times [f_e(\boldsymbol{v}_e)]\,d^3v_e$$

For the neutral gas is at rest ($\boldsymbol{u}_o = 0$) $\quad \langle \nu_I\rangle = n_{ao}\,n_{eo} \int_{V_I}^{+\infty} \sigma_I(|\boldsymbol{v}_e|)\;|\boldsymbol{v}_e|\,f_e(\boldsymbol{v}_e)\,d^3v_e$

Ionization in a fluid model

- The source term in the continuity equation is the average $\langle \nu_I \rangle = S_e$ representing the number of ionizing collisions by time and volume units.

$$\begin{cases} S_e = \langle \nu_I \rangle = n_{ao}\, n_{eo} \displaystyle\int_{V_I}^{+\infty} \sigma_I(|\boldsymbol{v}_e|)\; |\boldsymbol{v}_e|\, f_e(\boldsymbol{v}_e)\, d^3 v_e \\ S_e = \langle \nu_I \rangle = K_I\, n_{ao}\, n_{eo} = Q\, n_{eo} \end{cases}$$

Approximations:

Assuming a piecewise ionization cross section:

$$\sigma_I(v_e) = \begin{cases} \sigma_o & \text{if } |v_e| \geq V_I \\ 0 & \text{if } |v_e| < V_I \end{cases} \qquad S_e \sim Q \times n_{eo} = n_{eo} n_{ao}\, \sigma_o \int_{V_I}^{+\infty} |\boldsymbol{v}_e|\, f_e(\boldsymbol{v}_e)\, d^3 v_e$$

Sometimes it is also assumed:

$$\int_{V_I}^{+\infty} |\boldsymbol{v}_e|\, f_e(\boldsymbol{v}_e)\, d^3 v_e \sim V_{th,e} = \sqrt{\frac{8\, k_B T_e}{\pi m_e}} \qquad S_e \sim n_{eo} n_{ao}\, \sigma_o\, V_{th,e}$$

When the plasma is weakly ionized ($n_{ao} \gg n_e$) and not flowing ($\boldsymbol{u}_e = \boldsymbol{u}_i = 0$)

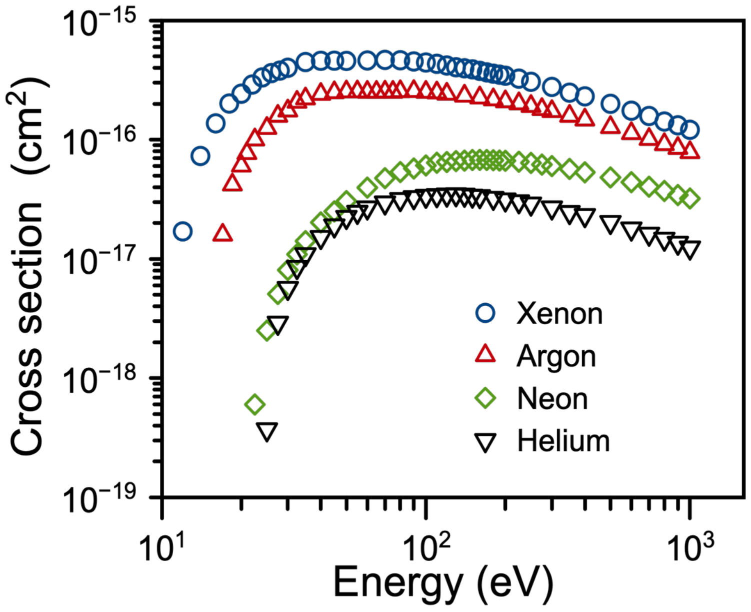
$$\left.\begin{aligned} \frac{\partial n_e}{\partial t} + \nabla \cdot (n_e\, \boldsymbol{u}_e) = K_I\, n_{ao}\, n_e \rightarrow \frac{\partial n_e}{\partial t} = Q\, n_e \\ \frac{\partial n_i}{\partial t} + \nabla \cdot (n_i\, \boldsymbol{u}_i) = K_I\, n_{ao}\, n_e \rightarrow \frac{\partial n_i}{\partial t} = Q\, n_e \end{aligned}\right\} \quad n_e(t) = e^{Qt}$$

Charge production is exponential!

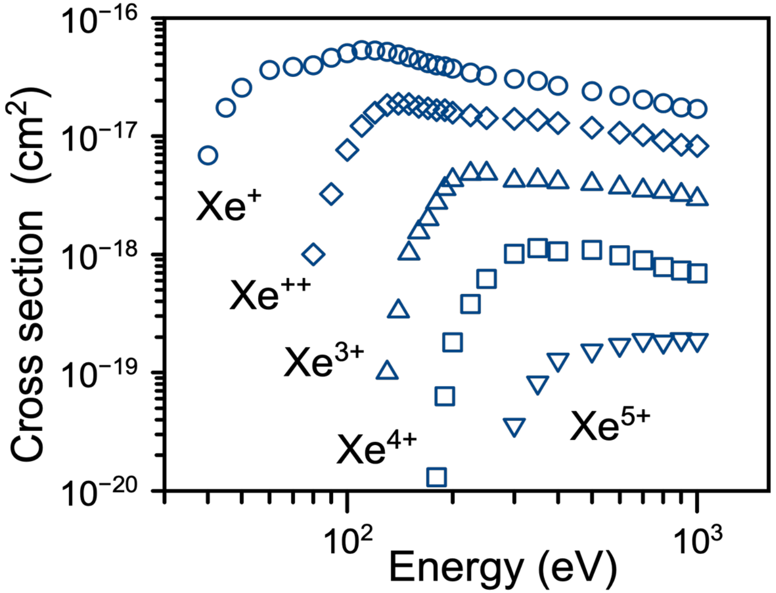
Electron impact ionization

- Electrons are produced by electron impact ionization of neutrals: $e^- + A \rightarrow A^+ + 2\,e^-$
- The ionizing electron needs energy over the ionization energy threshold E_I, (typically of tens of eV $(15.76\text{ eV for Argon})$
- The typical cross sections for the first ionization level of noble gases are $\sigma_I \sim (5-50)\times 10^{-16}\text{ cm}^2$

Cross sections for noble gases



Cross sections for different ionization levels of Xe



Estimation:

- The mean free path for ionizing collisions is, $$\lambda_I = \frac{1}{\sigma_I\, n_a}$$
- The number of ionizing collisions for one electron with velocity $v_e > \sqrt{2E_I/m_e}$ is, $$\nu_I = \frac{v_e}{\lambda_I} = \frac{1}{\tau_c} = \sigma_I\, n_a v_e$$

For n_e electrons

$$S_e = \nu_I \times n_e = \sigma_I\, n_a\, n_e\, v_e$$

Represents the number of ionization events per volume by time unit

Saha equation for a plasma in equilibrium

Developed by Meghnad Saha in 1920 for stellar plasmas.

CONDITION: Ionization comes from collisions of thermal atoms.

$$\frac{n_{i+1} n_e}{n_i} = \frac{2}{\lambda_{th}^3} \frac{g_{i+1}}{g_i} \exp\left[-\frac{\epsilon_{i+1} - \epsilon_i}{kT}\right]$$

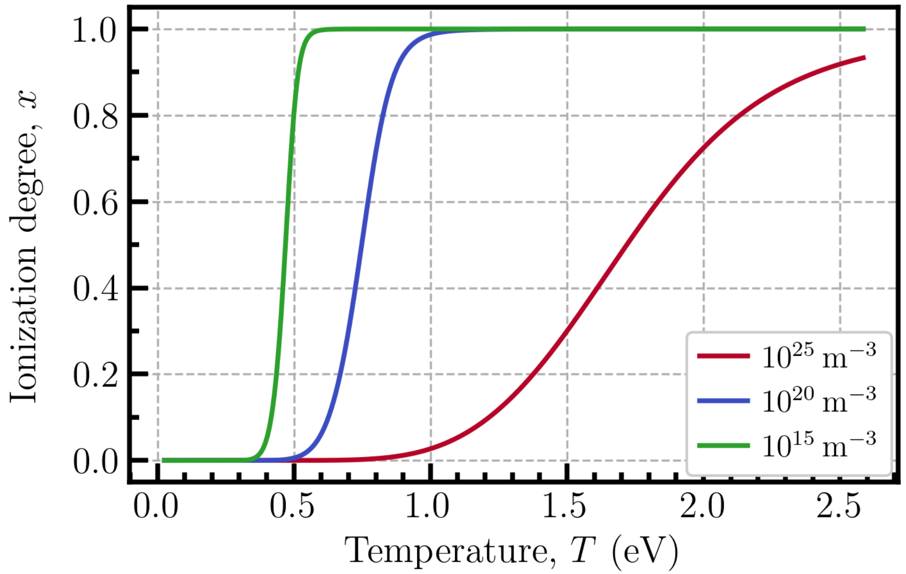
- n_i, n_e are the ion/electron density
- g_i s the degree of degeneracy of the i state
- ϵ_i is the ionization energy
- kT is the temperature (in energy units)
- λ_{th} is the thermal de Broglie wavelength

For a single species plasma, n_0, n_1, n_e (H, H^+, e):

$$\lambda_{th} = \frac{h}{\sqrt{2\pi m_e kT}}$$

$$n_1 = n_e;\ n = n_0 + n_1;\ g_1 = g_0;\ \frac{n_e^2}{n - n_e} = \frac{2}{\lambda_{th}^3} \exp\left[-\frac{\epsilon}{kT}\right]$$

$$x = \frac{n_1}{n} = \frac{n_e}{n};\ \frac{x^2}{1 - x} = \frac{2}{n\lambda_{th}^3} \exp\left[-\frac{\epsilon}{kT}\right]$$



$\epsilon = 13.6\,\mathrm{eV}$ **How is it possible?**

Why x increases at lower n ?

Momentum transport equation

- This equation of motion states the time evolution of momentum for each fluid element is due to self-consistent electromagnetic forces, pressure, shear and collisional interaction, as well as the creation/recombination of particles.

Charge and current density

$$\rho_{e\alpha} = q_\alpha\, n_\alpha$$

$$\boldsymbol{J}_{e\alpha} = \rho_{e\alpha}\, \boldsymbol{u}_\alpha = n_{e\alpha}\, q_\alpha\, \boldsymbol{u}_\alpha$$

Mass density: $\rho_{m\alpha} = m_\alpha\, n_\alpha$

Scalar (isotropic) pressure: $p_\alpha = n_\alpha\, k_B T_\alpha$

$$\rho_{m\alpha}\left(\frac{\partial \boldsymbol{u}_\alpha}{\partial t} + (\boldsymbol{u}_\alpha \cdot \nabla)\, \boldsymbol{u}_\alpha\right) = -\nabla p_\alpha - \nabla \cdot \boldsymbol{\Pi}_\alpha - m_\alpha \boldsymbol{u}_\alpha (S_\alpha - L_\alpha) + (\rho_{e\alpha}\ \boldsymbol{E} + \boldsymbol{J}_{e\alpha} \wedge \boldsymbol{B}) + \boldsymbol{R}_\alpha$$

- $\nabla \cdot \boldsymbol{\Pi}_\alpha$: Non-diagonal components of the stress tensor (usually neglected)
- $m_\alpha \boldsymbol{u}_\alpha (S_\alpha - L_\alpha)$: Momentum loss/gain by ionization/recombination
- $(\rho_{e\alpha}\ \boldsymbol{E} + \boldsymbol{J}_{e\alpha} \wedge \boldsymbol{B})$: Self-consistent electromagnetic force
- $\boldsymbol{R}_\alpha$: Friction force

- The friction force between species accounts for the collisional interaction between electrons, ions and neutrals.

$$\boldsymbol{R}_{\alpha\alpha} = 0 \quad \boldsymbol{R}_\alpha = \sum_{\beta \neq \alpha} \boldsymbol{R}_{\alpha\beta} \quad \boldsymbol{R}_{\alpha\beta} = -\boldsymbol{R}_{\beta\alpha} \quad \sum_\alpha \boldsymbol{R}_\alpha = 0$$

$$\boldsymbol{R}_{\alpha\beta} = -\mu_{\alpha\beta}\, n_\alpha\, \nu_{\alpha\beta} (\boldsymbol{u}_\alpha - \boldsymbol{u}_\beta)$$

$$\mu_{\alpha\beta} = \frac{m_\alpha\, m_\beta}{m_\alpha + m_\beta}, \qquad \mu_{ea} \simeq m_e$$

Where ν_{ea} is the collision frequency

The electromagnetic fields $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ are solutions of the Maxwell equations

$$\nabla \cdot \boldsymbol{E} = \frac{e}{\varepsilon_o}(n_i - n_e) \quad \nabla \wedge \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad \boldsymbol{J}_c = \sum_\alpha \boldsymbol{J}_{e\alpha}$$

$$\nabla \cdot \boldsymbol{B} = 0 \quad \nabla \wedge \boldsymbol{B} = \mu_o \boldsymbol{J}_c + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}$$

Electron models: isothermal (Boltzmann)

$$m_e \left(\frac{\partial \mathbf{u}_e}{\partial t} + \mathbf{u}_e \cdot \nabla \mathbf{u}_e \right) = e \nabla \phi - \frac{1}{n_e} \nabla p_e$$

$$\mathbf{u}_e \to 0; \; p_e = n_e k T_e; \; T_e = T_{e0}$$

$$e \nabla \phi = \frac{k T_{e0}}{n_e} \nabla n_e \to n_e = n_{e0} \exp \left(\frac{\phi - \phi_0}{k T_{e0}} \right)$$ n_{e0} is the density at ϕ_0

What are the consequences of this model?

- Electrons at constant temperature → Instant thermalization → infinite number of collisions
- Not cooling down when density reduces (plasma expansion)
- Electrons instantaneously response to changes in electric field

Electron models: polytropic

$$m_e\left(\frac{\partial \mathbf{u}_e}{\partial t}+\mathbf{u}_e\cdot\nabla\mathbf{u}_e\right)=e\nabla\phi-\frac{1}{n_e}\nabla p_e$$

$$\mathbf{u}_e\to 0;\ p_e=n_e kT_e;\ \frac{T_e}{T_{e0}}=\left(\frac{n_e}{n_{e0}}\right)^{\gamma_e-1}$$

$$e\nabla\phi=\frac{k}{n_e}\nabla n_e T_e\to n_e=n_{e0}\left[1+\frac{\gamma_e-1}{\gamma_e}\frac{e\left(\phi-\phi_0\right)}{kT_{e0}}\right]^{\frac{1}{\gamma_e-1}}$$

n_{e0} and T_{e0} are the density and temperature at ϕ_0

What are the consequences of this model?

- New unknown in the system $1<\gamma_e<5/3$
- Limits between isothermal and adiabatic descriptions
- Electrons instantaneously response to changes in electric field

These models can also be applied to ions