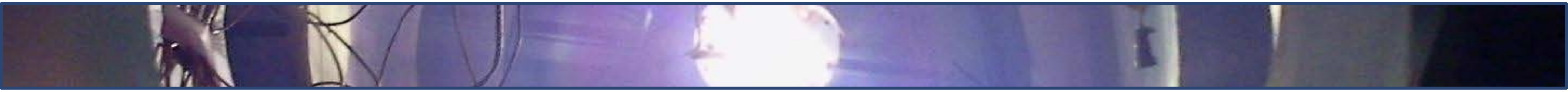

Máster Universitario en Ingeniería Aeronáutica

The Space Environment

Collisionless plasma approximation

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Maretial y Página web de la Asignatura basada en la web personal del Prof. Dr. L. Conde:
https://plasmalab.aero.upm.es/~lcl/EntornoEspacial/

Kinetic Theory: general and basic

For the species density in a point of the configuration space $\mathbf{r}$, as for the density of $n\,d\mathbf{r}$, in velocity space it is defined a similar number density being proportional to volume element $d\mathbf{r}$ and function *of* $\mathbf{r}$, t and $\mathbf{v}$ *as* $f(\mathbf{r}, \mathbf{v}, t)\,d\mathbf{r}$.

The number of particles *dN* in the volume d$\boldsymbol{r}$ with velocities lying between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$ is

$$dN = f(\mathbf{r}, \mathbf{v}, t)\,d\mathbf{r}\,d\mathbf{v} = f(\mathbf{r}, \mathbf{v}, t)\,d^3\mathrm{r}\,\mathrm{d}^3\mathrm{v}$$

Finally:

$f(\mathbf{r}, \mathbf{v}, t)$ is **the spatial-*velocity distribution function,*** it can be also understood as a probability density of points in the 6-D ***r-v*** phase space.

Kinetic Theory: A very general formulation is possible knowing the time evolution of f , valid for Inhomogeneous , anisotropic and non-equilibrium plasmas. It is quite general (usually unaffordable) and practical approximate theories are needed

To do this, is important to distinguish interactions ***in/out*** *a Debye Sphere, to account for:*

microscopic fields *simulated by collisional effects dominate in a Debye Sphere,* ***whereas macroscopic*** *fields* **(contribution *out* of Debye Sphere)** *also enter as a response of the* ***collective effects****, also changing them, intrinsic in any plasma*

ALL PLASMA DESCRIPTIONS APPEAR IN PLASMA INVIROMENT SCENARIOS

f motion understood as a continuity Eq. in a 6-D phase space.

- If particles number does not change in a phase space small (differential) volume, we have (The number of particles inside a small volume commoving with the observer does not change in time, no particle appears or disappears) we have the **collisionless Vlasov Equation**:

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{q}} \bullet \left(\boldsymbol{U}_q\, f \right) = \frac{\partial f}{\partial t} + div\left(\boldsymbol{U}_q\, f \right) = 0\ ;\ \ \boldsymbol{U}_q = (\dot{\mathbf{r}}, \dot{\mathbf{v}})$$

- However, If particles number $f\, d\mathbf{q}$ *varies* (***instantaneous velocity changes, for example ¡¡ they should seem to appear or disappear from a small velocity volume, for a fixed r***) in a 6-D phase space elemental volume, for instance, due to particles interactions *in a shorter scale (inside a Debye length)* , the time rate of change of f is not zero **: Collisional Boltzmann Equation**

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot (\mathbf{v}\, f) + \frac{\partial}{\partial \mathbf{v}} \cdot \left(\frac{\mathbf{F}}{m} f \right) = \left(\frac{\partial f}{\partial t} \right)_{coll} \neq 0 \ \ ; \ \ f \equiv f_\alpha, \ \ \alpha = e, i, a$$

- The *r.h.s.* term means the balance of s**ource-sinks** of particles in the volume element: **collision** may effectively to produce this effect, understood as if a particle with velocity v dissapears and another one appears with velocity v’

Note: **<u>The distribution can be interpreted as a number density satisfying the continuity equation</u>**

But we must consider r and v as independent variables, even though for a particle moving in phase space we have r(t) and v(t), with initial conditions r0 and v0. f can be interpreted as a density of points that start from given initial conditions and begin to move (Liouville)

Collisionless plasma

- A simple heuristic derivation based upon the fact that the number of particles is **conserved** in a volume phase space element, *df / dt* =0 , (density of points is constant in a fixed volume an measured in a co-moving frame with it, with the particles "fluid") gives the **VLASOV Equation**

$$\begin{aligned}\frac{\mathrm{d} f(\mathbf{r},\mathbf{v},t)}{\mathrm{d}t} &= \frac{\partial f}{\partial t} + \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\cdot\frac{\partial f}{\partial \mathbf{v}} \\ &= \frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m}\cdot\frac{\partial f}{\partial \mathbf{v}} = 0\end{aligned}$$

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}}\cdot\left(\boldsymbol{v}\, f\right) + \frac{\partial}{\partial \boldsymbol{v}}\cdot\left(\frac{q(\mathbf{E}+\mathbf{v}\times\mathbf{B})+m\mathbf{g}}{m}\, f\right) = 0,$$

since **v** and **r** are independent variables, the acceleration is **F**(**r**,**v**,t) / m for each species.

*The force **F** comes from average electromagnetic fields at mesoscopic scale (no microscopic fluctuating fields enter now in the model of average probablity density f)*

Approximation of neglecting collisions implies: *Scale still small inside a Debye sphere to consider that no particle-particle interaction occurs **mean-free-path large if compared with Debye length** , as in many plasma scenarios (astrophysical, lab …)*

- If **F** is a velocity divergence-free vector, it can be rewritten as : (observe notations)

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}}\cdot\left(\boldsymbol{v}\, f\right) + \frac{\partial}{\partial \boldsymbol{v}}\cdot\left(\frac{\boldsymbol{F}}{m}\, f\right) = 0, \quad \text{if} \quad \mathit{div}_{v}\boldsymbol{F} = \frac{\partial}{\partial \boldsymbol{v}}\cdot\boldsymbol{F}(\boldsymbol{r},\boldsymbol{v},t) = 0$$

$$\text{also used}: \frac{\partial}{\partial \boldsymbol{r}} \equiv \nabla_{r} \quad \text{and} \quad \frac{\partial}{\partial \boldsymbol{v}} \equiv \nabla_{v}$$

Resume

It is important to distinguish interactions ***in/out*** *a Debye Sphere,* ***in collisionless approximation*** *, microscopic fields are not relevant to control collective plasma behavior* ***whereas macroscopic*** *fields* **(contribution *out* of Debye Sphere)** *enter controlling all the plasma dynamics that evolves in a collective way thanks to the* ***effects driven by the fields (E, B, g ..): The field forces are responsible of collective effects.***

A (usually differential) equation is needed to get the evolution and steady states of $f(\mathbf{v},\mathbf{r},t)$ *:*

a *KINETIC EQUATION with/whithout collision terms*

We should be ***able to obtain the electromagnetic*** *forces from the plasma species distributions (to derive macroscopic magnitudes)* ***but***
they depend themselves on these forces ¡¡¡¡¡

Distribution f meaning: As said, it is considered the number of particles in a 6-D point space $(\mathbf{r}, \mathbf{v})$ **in a volume** $d\mathbf{r}\, d\mathbf{v}$ as

$$dN = f(\mathbf{r}, \mathbf{v}, t)\, d\mathbf{r}\, d\mathbf{v} = f(\mathbf{r}, \mathbf{v}, t)\, d^3\mathrm{r}\, \mathrm{d}^3\mathrm{v}$$

Any number $n(\mathbf{r}, t)\, d\mathbf{r}$ of particles in 3-D real spatial volume $d\mathbf{r}$ is set in a point of velocity space $\mathbf{v}$ as it were "*a particle*" in the volume $d\mathbf{r}$ at instant *t.*

Collisionless approximation. General Solution.

- Hence, the **distribution function** is related to the **description of the individual** deterministic **motion** equation of a test particle, this leads to the stament:

Any function of the constants of motion is a solution of the Vlasov equation.

- This can be stated by solving the equation with ***the method of Characteristic curves***, or testing this result by **simple sustitution**, as follows.

- From Mechanics: In particular, if the **Newtonian motion** equation is solved, we´ll have six constants of motion (the position and velocity initial conditions, for instance) leading to the solution $(\mathrm{d}f/\mathrm{d}t = 0)$ in the form:

$$f = f(c_1, c_2, \ldots, c_6)$$

$$\boxed{\sum_{i=1}^{6} \frac{\partial f}{\partial c_i}\left(\frac{\partial c_i}{\partial t} + \mathbf{v}\cdot\frac{\partial c_i}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m}\cdot\frac{\partial c_i}{\partial \mathbf{v}}\right) = \sum_{i=1}^{6} \frac{\partial f}{\partial c_i}\frac{\mathrm{d}c_i}{\mathrm{d}t} = 0}$$

For example *if only conservative forces are implied*, the mechanical energy E is a constant of classical motion, $C1 = E$ and any $f(E)$ is a solution since $dE/dt=0$ Prove it¡

Collisionless. A resolution method

- Resume: **Jeans´ Theorem** (first derived for stellar dynamics):

Any arbitrary function of the classical constants of motions is a solution of the Vlasov Equation

- In particular, if we have the solution of the Newtonian problem

$$\boldsymbol{F} = m\frac{d^2\boldsymbol{r}}{dt^2} \quad \textit{with} \quad \boldsymbol{r}(t_0)=\boldsymbol{r}_0 \quad \textit{and} \quad \boldsymbol{v}(t_0)=\boldsymbol{v}_0 \quad \Rightarrow \qquad \begin{matrix} \boldsymbol{r} = \boldsymbol{R}(t_0,\boldsymbol{r}_0,\boldsymbol{v}_0\,|\,t) \\ \boldsymbol{v} = \boldsymbol{V}(t_0,\boldsymbol{r}_0,\boldsymbol{v}_0\,|\,t) \end{matrix}$$

- The initial position and velocity conditions is a set **of six constants** of motion. Solving for the initial conditions (simply, by time inversión with the same funcional form):

$$\begin{matrix} \boldsymbol{r}_0 = \boldsymbol{R}(t,\boldsymbol{r},\boldsymbol{v}\,|\,t_0) \\ \boldsymbol{v}_0 = \boldsymbol{V}(t,\boldsymbol{r},\boldsymbol{v}\,|\,t_0) \end{matrix}$$

gives ,

$$\boxed{f = f_0(\boldsymbol{R}(t,\boldsymbol{r},\boldsymbol{v}\,|\,t_0),\boldsymbol{V}(t,\boldsymbol{r},\boldsymbol{v}\,|\,t_0)), \quad if \;\; f(\boldsymbol{r},\boldsymbol{v},\boldsymbol{t}=t_0) = f_0(\boldsymbol{r},\boldsymbol{v})}$$

Problem 0) Prove thar the Maxwell-Boltzmann distribution is a solution of the collisionless (Vlasov) plasma equation (consider for simplicity the 1-dim case o particles in a known electrostatic potential) see problem 2 below

Example (verify by simple substitution)

Problem 1) Verify the results: If the kinetic collisionless equation with a viscous friction force reads:

$$\frac{\partial}{\partial t} f(x,v,t) + \frac{\partial}{\partial x} v\, f + a \frac{\partial}{\partial v} f = 0 \quad with \quad a = -\gamma\, v$$

The deterministic solutions are :

$$v = v_0 \exp(-\gamma t)\ , \quad x = x_0 + \frac{v_0}{\gamma}\big(1 - \exp(-\gamma t)\ \big) \Rightarrow$$

$$v_0 = v \exp(\ \gamma t)\ , \quad x_0 = x + \frac{v}{\gamma}\big(1 - \exp(\gamma t)\ \big)$$

And the Vlasov Eq. admits the solution:

$$f = \Psi(x_0, v_0) = \Psi(x + \frac{v}{\gamma}\big(1 - e^{\gamma t}\ \big)\ ,\ \ v\, e^{\gamma t}\ )$$

But also, the solution can be expressed in terms of other constants of motion, i.e

$$f = \Phi(c_1, c_2) = \Phi(x + \frac{v}{\gamma}\ \ ,\ \ v\, e^{\gamma t}\ )$$

Resume. Advantages and drawbacks of Collisionless approx.

- The Vlasov equation gives time evolution of the distribution of each species by the action of average electromagnetic forces, disregarding the effect of collisional forces (interaction within the Debye sphere, microscopic fluctuating fields).

- The distribution *f* can be assumed as a density of points, it changes in *t* due to the flow of particles across the surface of an element of phase-space volume d**r**d**v**, the convective derivative is df/dt= 0 in the non-collisional approach, but it has the continuous change of particle density in configuration real space volume d**r** due to the action of external or self-generated forces (acceleration).
- Under the effects of an **electrostatic potential** a solution is the Maxwell-Boltzmann

$$f_e(\boldsymbol{r},\boldsymbol{v}) = A\exp\left(\frac{-mv^2/2+e\phi}{k_B T_e}\right) = A\exp\left(-E/k_B T_e\right)$$

- The Collisionless approximation describes collective effects dominated by the fields when collisional (particle interactions) are negligible.

- Limited usefulness: the mean free path of collisions is larger than the scale of the plasma under study , see Knudsen Numer test. The model is valid **for plasma waves** propagation analysis, short-term local effects or perturbative effects, with collective effects controlled by the fields.

- The Vlasov equation is invariant under r-R, v-V translations, Galilean transformation and time inversion, this means that **it does not change the entropy,** so, it does not describe irreversible processes of nature that would lead the system to an equilibrium or unique stationary *f* (one could find several steady state solutions)

Práctica

Problem 2) Find (and discuss the results) the “stationary” (time-independent) solution of :

$$\frac{\partial}{\partial t} f(x,v,t) + \frac{\partial}{\partial x} v\, f + \frac{\partial}{\partial v}\left(-\frac{d\phi(x)}{dx}\, f \right) = 0$$

Problem 3) Solve and discuss the simple equation

$$\frac{\partial}{\partial t} f(x,v,t) + \frac{\partial}{\partial x} v\, f + a\frac{\partial}{\partial v} f = 0$$

describing a distribution of particles in a uniform field in the cases:

A) In spatial uniform conditions.
B) Velocity uniform conditions.
C) As a Cauchy (initial value) problem in 1+1 phase-space.

Verify that *f is constant along the classical orbit, trajectories, of a particle. Extend this result to 3D+3D case.*

Hint: Apply Jean´s Th.

Note: as a first-order equation in partial derivatives, it can be also solved (formally) by the so-called Method of characteristic curves.
See http://en.wikipedia.org/wiki/Method_of_characteristics and references therein

Práctica

For the case of the Vlasov equation of Problem 1) consider the following two cases and verify thar there is no unique (stationary) solution for large *t*. Why?

a) Find the time dependent solutions for the following cases , and calculate the solution limit for large time t.

$$1)\ f(x,v,0)=n_0\sqrt{\frac{m}{2\pi kT}}\exp\left(-\frac{m}{2kT}(\gamma x+v)^2\right)$$

$$2)\ f(x,v,0)=n_0 e^{-\frac{m}{2kT}(\gamma x)^{\,2}}\sqrt{\frac{m}{2\pi kT}}\exp\left(-\frac{mv^2}{2kT}\right)$$

b) Calculate the number density *n(x,t)*, mean velocity *u(x,t)* and average kinetic energy *Ei(x,t)* at any time *t* and for the stationary cases.

Sol. For case 1) $n(x,t)=n_0\ \ ,\quad u(x,t)=-\gamma x,\left\langle \frac{1}{2}mv^2\right\rangle=\frac{1}{2}kT+\frac{1}{2}m(\gamma x)^2\rightarrow\left\langle\frac{1}{2}m(v-u)^2\right\rangle=\frac{1}{2}kT$

For case 2)

$$\textit{with}\ \ \beta=e^{\gamma t}\ \textit{and}\ \ \lambda^2=\beta^2+(1-\beta)^2:\begin{cases} n(x,t)=\dfrac{n_0}{\lambda}\exp(-\dfrac{m\beta^2\gamma^2x^2}{2kT\lambda^2}), \quad u(x,t)=-\gamma x\dfrac{\beta-1}{\lambda^2} \\ \left\langle\dfrac{1}{2}mv^2\right\rangle=\dfrac{1}{2\lambda^2}kT+\dfrac{1}{2}mu^2\rightarrow\left\langle\dfrac{1}{2}m(v-u)^2\right\rangle=\dfrac{1}{2\lambda^2}kT \end{cases}$$