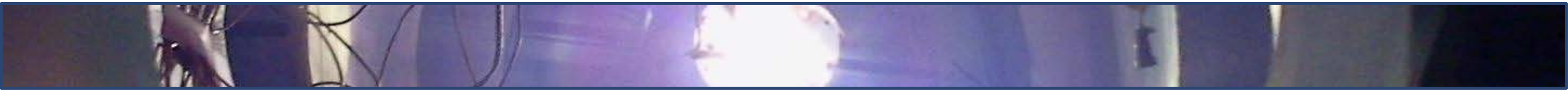

Máster Universitario en Ingeniería Aeronáutica

The Space Environment

Case Study Octuber 30th

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EL Entorno Espacial. MUIA. ***PLASMALAB-ETSIAE-UPM***.

Maretial y Página web de la Asignatura basada en la web personal del Prof. Dr. L. Conde:
https://plasmalab.aero.upm.es/~lcl/EntornoEspacial/

Extension: Information given by *f and* macroscopic magnitudes

- The **most important macroscopic measurables** magnitudes, for each plasma species α, derived from the distribution function are averages of velocity functions *H*(**v**) (usually powers of velocity components) in the form: (Definitions) α= electrons e. ions i. and neutrals a

$$\langle H\rangle_\alpha(\mathbf{r},t)=\frac{1}{n_\alpha}\int H(\mathbf{v},\mathbf{r},t)\,f_\alpha(\mathbf{v},\mathbf{r},t)\,d\mathbf{v}\Leftrightarrow n_\alpha\langle H\rangle_\alpha=\int H(\mathbf{v},\mathbf{r},t)\,f_\alpha(\mathbf{v},\mathbf{r},t)\,d\mathbf{v}$$

$$number\ density: n_\alpha=\int f_\alpha(\mathbf{v},\mathbf{r},t)\,d\mathbf{v}$$

$$Fluid\ velocity: \mathbf{u}_\alpha(\mathbf{r},t)=\langle\mathbf{v}\rangle=\frac{1}{n_\alpha}\int \mathbf{v}\,f_\alpha(\mathbf{v},\mathbf{r},t)\,d\mathbf{v}$$

$$Temperature: \frac{3}{2}k_BT_\alpha(\mathbf{r},t)=\left\langle\frac{1}{2}m_\alpha|\mathbf{v}-\mathbf{u}|^2\right\rangle=\frac{1}{n_\alpha}\int\frac{1}{2}m_\alpha|\mathbf{v}-\mathbf{u}|^2\,f_\alpha(\mathbf{v},\mathbf{r},t)\,d\mathbf{v}$$

- And the derived ones (observe that T and energy transport, heat, are measured accounting for thermal fluctuationa around fluid velocity ***u*** :

$$mass\ density: \rho=\sum_\alpha m_\alpha n_\alpha,\ \text{charged density}: \rho_q=\sum_\alpha q_\alpha n_\alpha$$

$$Particle\ flux: \boldsymbol{\Gamma}=\sum_\alpha\boldsymbol{\Gamma}_\alpha=\sum_\alpha n_\alpha\langle\mathbf{v}\rangle_\alpha$$

$$Electric\ current\ density: \mathbf{J}(\mathbf{r},t)=\sum_\alpha q_\alpha n_\alpha\langle\mathbf{v}\rangle_\alpha=\sum_\alpha q_\alpha n_\alpha\mathbf{u}_\alpha$$

$$Heat\ flux: \mathbf{q}(\mathbf{r},t)=n_\alpha\left\langle\frac{1}{2}m_\alpha|\mathbf{v}-\mathbf{u}|^2(\mathbf{v}-\mathbf{u})\right\rangle,\ Entropy: S=-k_B\iint f\ln f\,d\mathbf{v}\,d\mathbf{r}$$

Moments time evolution

- Finally, an effective and very general fluid equation is !!!!

$$\frac{\partial}{\partial t} n_\alpha \langle H \rangle + \frac{\partial}{\partial \mathbf{r}} \bullet \left[n_\alpha \langle \mathbf{v}\ H \rangle \right] - \frac{q_\alpha}{m_\alpha} n_\alpha \left\langle (\mathbf{E} + \mathbf{v}\times\mathbf{B}) \cdot \frac{\partial H}{\partial \mathbf{v}} \right\rangle =$$

$$\int d\mathbf{v} \left[H \left(\frac{\partial f_\alpha}{\partial t}\right)_c \right] \quad ; \ \text{for} \quad H = 1,\ v_i,\ \ v_i v_j,\ \ v_i v_j v_k,\ \ldots \ (i, j, k = x, y, z)$$

- A set of *scalar, vector or tensor equations* can be constructed from this. For example, without collision terms, for the zero- order moment *H*=1, we obtain the **continuity** equation for particle density, if the collision term is conservative (elastic):

$$\frac{\partial}{\partial t} n_\alpha + \frac{\partial}{\partial \mathbf{r}} \cdot [n_\alpha \boldsymbol{u}] = \int d\mathbf{v} \left(\frac{\partial f_\alpha}{\partial t}\right)_c = 0 \quad \text{(if no source-sink)}$$

and for *H*= **v** , it gives the fluid equation for **momentum** transport:

$$\frac{\partial}{\partial t} n_\alpha \boldsymbol{u} + \frac{\partial}{\partial \mathbf{r}} \cdot \left[n_\alpha \langle \mathbf{vv} \rangle \right] - \frac{q_\alpha}{m_\alpha} n_\alpha \langle (\mathbf{E} + \mathbf{v}\times\mathbf{B}) \bullet \mathrm{I} \rangle = 0,\ si\ \mathbf{v} = \boldsymbol{w} + \boldsymbol{u}(\boldsymbol{r})$$

$$\frac{\partial}{\partial t} n_\alpha m_\alpha \boldsymbol{u} + \frac{\partial}{\partial \mathbf{r}} \cdot \left[m_\alpha n_\alpha \boldsymbol{uu} + \underbrace{m_\alpha n_\alpha \langle \boldsymbol{ww} \rangle}_{pressure-stress-tensor} \right] = n_\alpha q_\alpha (\mathbf{E} + \boldsymbol{u}\times\mathbf{B})$$

and so on. E.g. the equation for **kinetic energy (2nd order)** involves pressure tensor components , **heat flux** (3rd order moment) , and the internal and electromagnetic energies. The velocity is usually decomposed into two parts: macroscopic **u** and thermal fluctuating **w** for calculations.

Example (verify by simple substitution)

Problem 1) Verify the results: If the kinetic collisionless equation with a viscous friction force reads:

$$\frac{\partial}{\partial t} f(x,v,t) + \frac{\partial}{\partial x} v\, f + a \frac{\partial}{\partial v} f = 0 \quad with \quad a = -\gamma\, v$$

The deterministic solutions are :

$$v = v_0 \exp(-\gamma t) \ , \quad x = x_0 + \frac{v_0}{\gamma}\left(1 - \exp(-\gamma t)\right) \Rightarrow$$

$$v_0 = v \exp(\ \gamma t) \ , \quad x_0 = x + \frac{v}{\gamma}\left(1 - \exp(\gamma t)\right)$$

And the Vlasov Eq. admits the solution:

$$f = \Psi(x_0, v_0) = \Psi(x + \frac{v}{\gamma}\left(1 - e^{\gamma t}\right) \ , \ v\, e^{\gamma t})$$

But also, the solution can be expressed in terms of other constants of motion, i.e

$$f = \Phi(c_1, c_2) = \Phi(x + \frac{v}{\gamma} \ , \ v\, e^{\gamma t})$$

Práctica

Problem 2) Find (and discuss the results) the “stationary” (time-independent) solution of :

$$\frac{\partial}{\partial t} f(x,v,t) + \frac{\partial}{\partial x} v\, f + \frac{\partial}{\partial v}\left(-\frac{d\phi(x)}{dx}\, f \right) = 0$$

Problem 3) Solve and discuss the simple equation

$$\frac{\partial}{\partial t} f(x,v,t) + \frac{\partial}{\partial x} v\, f + a \frac{\partial}{\partial v} f = 0$$

describing a distribution of particles in a uniform field in the cases:

A) In spatial uniform conditions.
B) Velocity uniform conditions.
C) As a Cauchy (initial value) problem in 1+1 phase-space.

Verify that *f is constant along the classical orbit, trajectories, of a particle. Extend this result to 3D+3D case.*

Hint: Apply Jean´s Th.

Note: as a first-order equation in partial derivatives, it can be also solved (formally) by the so-called Method of characteristic curves.
See http://en.wikipedia.org/wiki/Method_of_characteristics and references therein

Práctica

For the case of the Vlasov equation of Problem 1) consider the following two cases and verify thar there is no unique (stationary) solution for large *t*. Why?

a) Find the time dependent solutions for the following cases , and calculate the solution limit for large time t.

$$1)\ f(x,v,0) = n_0\sqrt{\frac{m}{2\pi kT}}\exp\left(-\frac{m}{2kT}(\gamma x+v)^2\right)$$

$$2)\ f(x,v,0) = n_0 e^{-\frac{m}{2kT}(\gamma x)^2}\sqrt{\frac{m}{2\pi kT}}\exp\left(-\frac{mv^2}{2kT}\right)$$

b) Calculate the number density *n(x,t)*, mean velocity *u(x,t)* and average kinetic energy *Ei(x,t)* at any time *t* and for the stationary cases.

Sol. For case 1) $n(x,t)=n_0\ ,\quad u(x,t)=-\gamma x, \left\langle \frac{1}{2}mv^2\right\rangle = \frac{1}{2}kT+\frac{1}{2}m(\gamma x)^2 \to \left\langle \frac{1}{2}m(v-u)^2\right\rangle = \frac{1}{2}kT$

For case 2)

$$\text{with } \beta = e^{\gamma t} \text{ and } \lambda^2 = \beta^2 + (1-\beta)^2 : \begin{cases} n(x,t) = \frac{n_0}{\lambda}\exp\left(-\frac{m\beta^2\gamma^2x^2}{2kT\lambda^2}\right), \quad u(x,t) = -\gamma x\frac{\beta-1}{\lambda^2} \\ \left\langle \frac{1}{2}mv^2\right\rangle = \frac{1}{2\lambda^2}kT+\frac{1}{2}mu^2 \to \left\langle \frac{1}{2}m(v-u)^2\right\rangle = \frac{1}{2\lambda^2}kT \end{cases}$$

Practica: Un ejemplo de término collisional (de Krook, BGK)

Problem 2) Consider the space 1-D homogeneous distribution governed by :

$$\frac{\partial}{\partial t} f(v,t) + a\frac{\partial}{\partial v} f = -\nu\,(f - f_0(v)) \ ; \ (a\ ,\nu\ are\ \text{constants})$$

Where f_0 is a Maxwellian distribution of zero mean velocity and constant density n_0 and temperature T_0, the acceleration a is constant
Discuss the meaning of each term.

a) Verify that there is a stationary solution *f(v)*. Does this steady state solution depend on the initial function *f(v,0)*? ¿what boundary conditions should satisfied f?

b) Find the transient and steady solutions for *a*=0 and for arbitrary *f(v,0)*.

c) Compute for this solution *n(t)*, *u(t)* and *T(t)*.

$$Sol.\ f(v,t) = f_0(v) + (f(v,0) - f_0(v))e^{-\nu t} \rightarrow$$
$$n(t) = n_0(1-e^{-\nu t}) + n(0)e^{-\nu t}$$
$$u(t) = n(0)u(0)e^{-\nu t}/n(t) \quad \text{and} \ \ T(t) = \ldots$$

Problem (a Fokker-Planck equation as a more realistic collison term)

4) Consider the 1D kinetic equation in the convection-diffusion form:

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial v}\left[-\gamma(v-u) - \frac{\partial}{\partial v}D\right]f(v,t) \equiv -\frac{\partial j}{\partial v}$$

where

$$\int_{v=-\infty}^{\infty} f(v,t)dv = n(t), \quad n(t)u(t) = \int_{v=-\infty}^{\infty} v\, f(v,t)dv$$

Assuming that j *(a density flow)*, f and its derivative with respecto to v vanish for large $|v|$,

a) Discuss the meaning of each term (dealing D and γ as contants) in order to consider this equation as a collision 1-D operator (**Fokker-Planck** equation).

b) Using integration by parts on both sides of the eq. verify that n(t) =n(0) and u(t) = u(0) are constant in time and find D to ensure that the temperature T is also a constant (collisional invariant).

c) Assume that a stationary solution in the form

$$\boxed{f_s(v) = C\exp(-a(v-u)^2)}$$

exits and find it.

c) Find a simple extension of this collision term for the 3-D velocity distribution function.

$$Sol.consider:\ \gamma = \nu_c \quad and \quad D = kT\nu_c/m$$

Application to transport: Perturbation method. Example

A method to compute the **transport coefficients** (as plasma conductivities) operates as follows: by **linearizing** this Eq. assuming **a small deviation f_1 from the isotropic f_0** steady state distribution with homogeneous density *n* and temperature *T* :

$$f(\mathbf{r},\mathbf{v},t) \simeq f_0(v) + f_1(\mathbf{r},\mathbf{v},t) \quad ; \quad |f_1| \ll f_0$$

Example:**Electrical conductivity**. Assuming spatially uniform $f_{1,}$ T and n, in a non-magnetized plasma (single species of **charge** *Ze*) in a ***small*** uniform field **E:**

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}}+\frac{q\mathbf{E}}{m}\cdot\frac{\partial}{\partial \mathbf{v}}\right)\underbrace{\left(f_0(v)+f_1(\mathbf{v})\right)}_{f} \simeq 0+\frac{eZ\mathbf{E}}{m}\cdot\frac{\partial f_0}{\partial \mathbf{v}} = -\nu_c\left(f_1 + f_0 - f_0\right), gives\ f_1\ (if\ \left|\mathbf{E}\ f_1\right| \to 0)$$

$$\sigma\mathbf{E} = \mathbf{j} = qn\mathbf{u} = Ze\int \mathbf{v} f\, d\mathbf{v} = Ze\int \mathbf{v} f_1 d\mathbf{v} = Ze\frac{Ze\mathbf{E}}{m\nu_c}\, n \quad , \ pues \int \mathbf{v} f_0 d\mathbf{v} = n\mathbf{u} = 0$$

$$\sigma = Z^2\frac{e^2 n}{m\nu_c} \ ; \ In\ general : \mathbf{j} = \sum_{\alpha} q_\alpha n_\alpha \mathbf{u}_\alpha \Rightarrow \sigma = \sum_{\alpha}\frac{Z_\alpha{}^2 e^2 n_\alpha}{m_\alpha \nu_{c\,\alpha}}$$

As greater the collision frequency is, the plasma is less conductive (is it real? The collision frequency rwould be proportional to 1/n ¡¡¡¡¡ . Really: This conductivity tends to a saturation value for large density n).

The heat flux **q** would lead to the thermoelectrical coefficient β:

$$\mathbf{q} = \int \frac{1}{2} m v^2\ \mathbf{v}\ f\, d\mathbf{v} = \int \frac{1}{2} m v^2\ \mathbf{v}\ f_1\, d\mathbf{v} = -\beta\mathbf{E}$$

Práctica

Problem 3) Find (and discuss the results) the “stationary” (time-independent) solution of :

$$\frac{\partial}{\partial t} f(x,v,t) + \frac{\partial}{\partial x} v\, f + \frac{\partial}{\partial v}\left(-\frac{d\phi(x)}{dx} f \right) = 0$$

Problem 2) Solve and discuss the simple equation

$$\frac{\partial}{\partial t} f(x,v,t) + \frac{\partial}{\partial x} v\, f + a \frac{\partial}{\partial v} f = 0$$

describing a distribution of particles in a uniform field in the cases:

A) In spatial uniform conditions.
B) Velocity uniform conditions.
C) As a Cauchy (initial value) problem in 1+1 phase-space.

Verify that *f is constant along the classical orbit of a particle. Extend this result to 3D+3D case.*

Hint: See http://en.wikipedia.org/wiki/Method_of_characteristics and references therein

Nota: como ecuación de primer orden en derivadas parciales , puede resolverse (formalemente) pro el llamado Método de las curvas características,.

Ejemplos

Problem 4) Verify the results: If the kinetic collisionless equation with a viscous friction force reads:

$$\frac{\partial}{\partial t} f(x,v,t) + \frac{\partial}{\partial x} v\, f + a \frac{\partial}{\partial v} f = 0 \quad with \quad a = -\gamma\, v$$

The deterministic solutions are :

$$v = v_0 \exp(-\gamma t)\ , \quad x = x_0 + \frac{v_0}{\gamma}\left(1 - \exp(-\gamma t)\,\right) \Rightarrow$$

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And the Vlasov Eq. admits the solution:

$$f = \Psi(x_0, v_0) = \Psi(x + \frac{v}{\gamma}\left(1 - e^{\gamma t}\,\right)\ ,\ \ v\, e^{\gamma t}\,)$$

But also, the solution can be expressed in terms of other constants of motion, i.e

$$f = \Phi(c_1, c_2) = \Phi(x + \frac{v}{\gamma}\ \ ,\ \ v\, e^{\gamma t}\,)$$

ejemplo

a) Find the time dependent solutions for the following cases , and calculate the solution limit for large time t.

$$1)\, f(x,v,0) = n_0 \sqrt{\frac{m}{2\pi kT}} \exp\left(-\frac{m}{2kT}(\gamma x + v)^2 \right)$$

$$2)\, f(x,v,0) = n_0 e^{-\frac{m}{2kT}(\gamma x)^2} \sqrt{\frac{m}{2\pi kT}} \exp\left(-\frac{mv^2}{2kT} \right)$$

b) Calculate the number density *n(x,t)*, mean velocity *u(x,t)* and average kinetic energy *Ei(x,t)* .

Sol. For case 1) $n(x,t) = n_0 \quad , \quad u(x,t) = -\gamma x, \left\langle \frac{1}{2} mv^2 \right\rangle = \frac{1}{2} kT + \frac{1}{2} m(\gamma x)^2 \rightarrow \left\langle \frac{1}{2} m(v-u)^2 \right\rangle = \frac{1}{2} kT$

For case 2)

$$with\ \beta = e^{\gamma t}\ and\ \lambda^2 = \beta^2 + (1-\beta)^2 : \begin{cases} n(x,t) = \dfrac{n_0}{\lambda} \exp(-\dfrac{m\beta^2\gamma^2x^2}{2kT\lambda^2}), \quad u(x,t) = -\gamma x \dfrac{\beta - 1}{\lambda^2} \\ \left\langle \dfrac{1}{2} mv^2 \right\rangle = \dfrac{1}{2\lambda^2} kT + \dfrac{1}{2} m u^2 \rightarrow \left\langle \dfrac{1}{2} m(v-u)^2 \right\rangle = \dfrac{1}{2\lambda^2} kT \end{cases}$$

Soluciones a cuestiones y problemas (guiones)

/ Vlasov . 1.

Si $\vec{F}$ es fuerza central,

$$\vec{F} = -\nabla E_p(r) = -\frac{\partial}{\partial \vec{r}} E_p(r) = -E'(r)\frac{\vec{r}}{r}$$

la ecuación para la distribución de la especie α es ($f_\alpha = f$)

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{r}} + \frac{\partial}{\partial \vec{v}}\cdot\left| -\frac{f}{m}\frac{\partial E_p}{\partial \vec{r}}\right| = 0 \qquad (1)$$

luego, cualquier función de la constante

$$E_0 = \frac{1}{2}mv^2 + E_p(r) = \text{Energía Mecánica}$$

es solución

$$f = \psi\left(\frac{1}{2}mv^2 + E_p(r)\right)$$

Verificación:

$$\frac{\partial f}{\partial x} = \frac{\partial \psi}{\partial E_0}\frac{\partial E_0}{\partial x} = \psi'(E_0)\frac{\partial E_p}{\partial r}\frac{x}{r}$$

luego → $\frac{\partial f}{\partial \vec{r}} = \psi'(E_0)\frac{\partial E_p}{\partial r}\frac{\vec{r}}{r} = \psi'(E_0)\nabla_r E_p$

y $\frac{\partial f}{\partial \vec{v}} = \psi'(E_0)\frac{1}{2}m2\vec{v}$

dando en (1)

$$0 + \psi'\left[\vec{v}\cdot\nabla_r E_p - m\frac{\vec{v}}{m}\cdot\nabla_r E_p\right] = 0$$

Ejm en plasma de corona solar E_p es

$$E_p = -\frac{G M_{sol} m}{r}$$

(El campo gravitatorio confina plasma)

Vlasov 3)

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x} v f + \frac{\partial}{\partial v}\left| -\frac{d\phi}{dx} f \right| = 0 \qquad (1)$$

Es el mismo caso anterior pero 1-D, luego

$$f = \psi\left(\frac{v^2}{2} + \phi\right) = \psi(E) \quad \left(E \ \frac{\text{Jul}}{\text{kg}}\right)$$

es solución independiente de t, que no "estacionaria" pues no se asegura que una solución transitoria $f(x,v,t)$ dé para t largo la forma anterior: La aproximación de Vlasov no es compatible con la irreversibilidad (aumento de entropía).

Comprobación

$$f = \psi(E)$$

$$\frac{\partial f}{\partial t} = 0, \qquad \frac{\partial f}{\partial x} = \psi'(E)\frac{d\phi}{dx}, \quad \frac{\partial f}{\partial v} = \psi'(E)\, v$$

luego (1) se cumple.

En particular si $\psi(E) = c\, e^{-\frac{E}{kT/m}}$

se tiene

$$f = c\, e^{-\frac{m}{kT}\left(\frac{1}{2}v^2 + \phi\right)}$$

es la distribución de Maxwell-Boltzmann

y

$$n(x) = \int_{-\infty}^{\infty} f\, dv = c\, e^{-\frac{m\phi}{kT}} \sqrt{\frac{2\pi kT}{m}}$$

es la distribución (espacial) de Boltzmann.

Problema 2)

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + a\frac{\partial f}{\partial v} = 0$$

Es un caso unidimensional (x,v) con aceleración determinista (partícula de referencia) constante, por ejm. electrones en ~~un~~ campo $\vec{E}$ uniforme

$$a = \frac{q_e E}{m_e} = -\frac{eE}{m}$$

La energía potencial de una fuerza constante da para el potencial de a:

$$\frac{E_p}{m} = \phi = -ax + \phi_0 \;;\; a = -\frac{\partial \phi}{\partial x}$$

Para resolverlo, la solución es cualquier función de las constantes del movimiento

$$f(x,v,t) = f(x_0, v_0)$$

como

$$\begin{cases} x = x_0 + v_0 t + \frac{1}{2}at^2 \\ v = v_0 + at \end{cases} \Rightarrow \begin{cases} x_0 = x - vt + \frac{1}{2}at^2 \\ v_0 = v - at \end{cases}$$

y

$$f(x,v,t) = f\left(x - vt + \frac{1}{2}at^2, v - at\right) = f_0(x_0, v_0)$$

por ejmplo, si

A) $f(x,v,0) = Ce^{-mv^2/2}$ (uniforme)

$$f = Ce^{-m(v-at)^2/2}$$

es solución.

luego si $\frac{\partial f}{\partial x} = 0 \Rightarrow f = f(v - at)$

B/ si $\frac{\partial f}{\partial v} = 0$

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0 \Rightarrow \quad dt = \frac{dx}{v}, \quad v = v_0$$

$$x - V_0 t = cte.$$

se tiene

$$f = \varphi(x - vt) = f_0(x - vt)$$

solución que viaja por eje $\overline{ox}^+$ con velocidad v sin modificar su forma.

C/ Conocida $f(x, v, t=0) = f_0(x, v)$ se tiene, en general:

$$f(x, v, t) = f_0\left(x - vt + \frac{1}{2}at^2, v - at\right)$$

que, para $t \to \infty$ no tiene porque ser la solución independiente del tiempo

$$f_s = \Phi\left(\frac{1}{2}v^2 + ax\right) = \Phi\left(\frac{E_p}{m}\right)$$

como dije antes.

En 3 dimensiones, sería igual si $\vec{a}$ es cte

$$f(\vec{r}, \vec{v}, t) = f_0\left(\vec{r} - \vec{v}t + \frac{1}{2}\vec{a}t^2, \vec{v} - \vec{a}t\right)$$

Problem 4) ⑤

Ahora $a = -\gamma v$, luego

$$\begin{cases} a = \frac{dv}{dt} = -\gamma v \\ v = \frac{dx}{dt} \end{cases} \rightarrow \begin{cases} v = v_0 e^{-\gamma t} \\ x = x_0 + \frac{v_0}{\gamma}(1-e^{-\gamma t}) \end{cases}$$

y por tanto

$$\begin{cases} x_0 = x + \frac{v}{\gamma}(1-e^{\gamma t}) \\ v_0 = v e^{\gamma t} \end{cases} \qquad (1)$$

lo que da la solución

$$f(x,v,t) = f(x_0, v_0)$$

usando (1).

Se ve también que

$$s = x + \frac{v}{\gamma} = x_0 + \frac{v_0}{\gamma} \quad (\forall t),\ \frac{ds}{dt} = 0$$

es una constante del movimiento también, luego puede ponerse

$$f(x,v,t) = \Phi\left(x+\frac{v}{\gamma},\ x+\frac{v}{\gamma}(1-e^{\gamma t})\right)$$

entre otras posibilidades.

Casos particulares :

$$1)\ f(x,v,0) = n_0\sqrt{\frac{m}{2\pi kT}}\, e^{-\frac{m}{2kT}(v+\gamma x)^2}$$

cambiando x y v por $x_0(t)$ y $v_0(t)$ de 1 se ve que

$$f(x,v,t) = f(x,v,0)$$

pues $v+\gamma x$ no cambia con t, tampoco f, que resulta independiente del tiempo.

Las variables fluido dinámicas se obtienen integrando (salen integrales gausianas)
Basta usar integrales del tipo

$$I(a) = \int_{-\infty}^{\infty} z^n e^{-\frac{z^2}{a^2}}$$

o, general, $\displaystyle\int_0^{\infty} z^n e^{-az^2} dz = \frac{\Gamma\left(\frac{n+1}{2}\right)}{(2a)^{\frac{n+1}{2}}}$ $(n > -1, a > 0)$

que salen en tablas.

Por ejm.

$$n(x) = \int_{-\infty}^{\infty} f\, dv = \int_{-\infty}^{\infty} n_0 \sqrt{\frac{m}{2\pi kT}}\, e^{-\frac{m}{2kT}(v+\gamma x)^2} dv$$

con $w = v + \gamma x$, $dw = dv$

$$n(x) = \int_{-\infty}^{\infty} n_0 \sqrt{\frac{m}{2\pi kT}}\, e^{-\frac{m}{2kT}w^2} dw = n_0$$

para $\langle v \rangle = u$

$$nu = \int_{-\infty}^{\infty} v f\, dv = n_0 \int_{-\infty}^{\infty} (w - \gamma x) \sqrt{\frac{m}{2\pi kT}}\, e^{-\frac{mw^2}{2kT}} dw$$

$$= 0 - \gamma x \Rightarrow \boxed{-\gamma x = u}$$

y $\displaystyle n\langle \tfrac{1}{2} m v^2 \rangle = \int_{-\infty}^{\infty} \frac{1}{2} m (w - \gamma x)^2 f\, dw = \frac{1}{2} kT + \frac{1}{2} m (\gamma x)^2$

que son independientes del tiempo.

2) Para este caso,

$$f(x,v,t) = n_0 \exp\left| -\frac{m\gamma^2}{2kT} x_0^2(t) - \frac{m}{2kT} v_0^2(t) \right| \sqrt{\frac{m}{2\pi kT}}$$

con x_0 y v_0 de (1).

Se ve ahora que si $t \to \infty$ f tiende a cero, no a la solución independiente de t anterior (reversibilidad de Vlasov Eq.).

El cálculo de n, u y T lleva integrales similares al caso anterior, mejor usar un programa de integración simbólica como Maxima, Mathematica, Derive ...

$$n = \int_{-\infty}^{\infty} f \, dv = \frac{n_0}{\lambda} e^{-\frac{m\beta^2\gamma^2x^2}{2kT\lambda^2}} = n(x,t)$$

$$u = \frac{\int_{-\infty}^{\infty} v f \, dv}{n} = -\gamma x \frac{\beta - 1}{\lambda^2}$$

y

$$\left\langle \frac{1}{2} m (v-u)^2 \right\rangle = \frac{1}{2} k T_{emp} =$$

$$= \frac{1}{2\lambda^2} kT \Rightarrow T_{emp.} = \frac{T}{\lambda^2}$$

con

$$\beta = e^{\gamma t}, \quad \lambda^2 = \beta^2 + (1-\beta)^2$$

se ve que

$$\lim_{t\to\infty} n(t) = \frac{n_0}{\lambda} e^{\frac{m\gamma^2x^2}{4kT}}$$

$$\lim_{t\to\infty} T_{em} = 0$$

etc

$$\lim_{t\to\infty} u = 0$$

Un ejemplo de término collisional (de Krook, BGK)

Problem 2) Consider the space 1-D homogeneous distribution governed by :

$$\frac{\partial}{\partial t} f(v,t) + a\frac{\partial}{\partial v} f = -\nu\left(f - f_0(v)\right) \ ; \ (a \ , \nu \ are \text{ constants})$$

Where f_0 is a Maxwellian distribution of zero mean velocity and constant density n_0 and temperature T_0. Discuss the meaning of each term.

a) Verify that there is a stationary solution *f(v)*. Does this steady state solution depend on the initial function *f(v,0)*? ¿what the boundary condition should satisfied f?

b) Find the transient and steady solutions for a=0 and an arbitrary *f(v,0)*.

c) Compute for this solution *n(t)*, *u(t)* and *T(t)*.

$$Sol.\ f(v,t) = f_0(v) + (f(v,0) - f_0(v))e^{-\nu t} \rightarrow$$

$$n(t) = n_0(1 - e^{-\nu t}) + n(0)e^{-\nu t}$$

$$u(t) = n(0)u(0)e^{-\nu t} / n(t) \quad \text{and} \quad T(t) = \ldots$$

/ Colisiones , K3.

Problema 1

$$\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial v} = -\nu\left(f - f_0(v)\right)$$

con a y ν constantes ; f uniforme en x.

$a = \frac{F}{m}$ → partículas en campo uniforme (efecto colectivo)

$-\nu f$ → pérdida de partículas por colisiones

$+\nu f_0$ → ganancia de partículas

El término $-\nu(f - f_0)$ tiende a restaurar los cambios de f hacia f_0, (Maxwelliana referente)

$$f_0 = n_0\sqrt{\frac{m}{2\pi KT}}\,\exp\left\{-\frac{mv^2}{2KT}\right\}$$

a) Si $\frac{\partial f}{\partial t} = 0$ queda

$$a\frac{df}{dv} = -\nu f + \nu f_0(v)$$

que es una ecuación diferencial ordinaria lineal en 1-D, fácil de resolver, dando la solución estacionaria $f_s(v)$ independiente de condición inicial.

Si $a = 0$, entonces $f_s = f_0(v)$

b) $a = 0$, $\frac{\partial f}{\partial t} = -\nu(f - f_0) \Rightarrow \frac{df}{f - f_0} = -\nu\, dt$

con v fijo, da

$$\ln(f - f_0) = -\nu t + cte \Rightarrow$$

$$f(v,t) = f_0(v) + C(v)\,e^{-\nu t}$$

con la condición inicial $f(v,0)$ se tiene

$$f(v,0) = f_0(v) + C(v)\cdot 1$$

luego

$$f(v,t) = f_0(v) + \left(f(v,0) - f_0(v)\right)e^{-\nu t}$$

solución transitoria, con $f(v, t\to\infty) = f_0(v)$.

c/ Momentos de $f(v,t)$:

$$n = \int_{-\infty}^{\infty} f\,dv = \int \left[f_0 + \left(f(v,0) - f_0\right)e^{-\nu t}\right]dv$$

$$= n_0 + \left[n(0) - n_0\right]e^{-\nu t} \xrightarrow[t\to\infty]{} n_0$$

$$nu = \int_{-\infty}^{\infty} v f\,dv = \int v\left[f_0 + \left(f(v,0) - f_0\right)e^{-\nu t}\right]dv$$

$$= n_0\cdot 0 + \left(n(0)\,u(0) - n_0\cdot 0\right)e^{-\nu t}$$

$$u(t) = \frac{u(0)\,n(0)\,e^{-\nu t}}{n_0 + \left[n(0) - n_0\right]e^{-\nu t}} \xrightarrow[t\to\infty]{} 0$$

y la energía

$$\frac{1}{2} n k T' = \int \frac{1}{2} m v^2 f\,dv = n_0 \frac{1}{2} k T + \left[\frac{n(0)}{2} k T(0) - \frac{1}{2} k T n_0\right]e^{-\nu t}$$

$$T(t) = T' = \frac{n_0 T + \left(n(0)\,T(0) - n_0 T\right)e^{-\nu t}}{n_0 + \left(n(0) - n_0\right)e^{-\nu t}} \xrightarrow[t\to\infty]{} T$$

Si se elige una $f(v,0)$ inicial tal que $n(0) = n_0$, $u(0) = 0$ y $T(0) = T$ entonces

$$\left.\begin{aligned} n(t) &= n_0 \\ u(t) &= 0 \\ T(t) &= T \end{aligned}\right\} \text{ son constantes en } t$$

Application to transport: Perturbation method. Examples

A method to compute the **transport coefficients** (as plasma conductivities) operates as follows: by **linearizing** this Eq. assuming **a small deviation f_1 from the isotropic f_0** steady state distribution :

$$f(\mathbf{r},\mathbf{v},t) \simeq f_0(v) + f_1(\mathbf{r},\mathbf{v},t) \quad ; \quad |f_1| \ll f_0$$

Example:**Electrical conductivity**. Assuming spatially uniform f_1, T and n, in a non-magnetized plasma (single species of **charge** Ze) in a ***small*** uniform field **E:** (f1 E es infinitésimo de orden 2, despreciable).

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}} + \frac{q\mathbf{E}}{m}\cdot\frac{\partial}{\partial \mathbf{v}}\right)\underbrace{\left(f_0(v) + f_1(\mathbf{v})\right)}_{f} \simeq 0 + \frac{eZ\mathbf{E}}{m}\cdot\frac{\partial f_0}{\partial \mathbf{v}} = -\nu_c(f_1 + f_0 - f_0), gives\ f_1 \ (if\ \ |\mathbf{E}\ f_1| \to 0)$$

$$\sigma\mathbf{E} = \mathbf{j} = qn\mathbf{u} = Ze\int \mathbf{v} f\, d\mathbf{v} = Ze\int \mathbf{v} f_1 d\mathbf{v} = Ze\frac{Ze\mathbf{E}}{m\nu_c} \quad , \ pues \int \mathbf{v} f_0 d\mathbf{v} = n\mathbf{u} = 0$$

$$\sigma = Z^2\frac{e^2 n}{m\nu_c} \ ; \ In\ general: \mathbf{j} = \sum_\alpha q_\alpha n_\alpha \mathbf{u}_\alpha \Rightarrow \sigma = \sum_\alpha \frac{Z_\alpha{}^2 e^2 n_\alpha}{m_\alpha \nu_{c\,\alpha}}$$

As greater the collision frequency is, the plasma is less conductive (is it real? The collision frequency rwould be proportional to 1/n ¡¡¡¡¡ . Really: This conductivity tends to a saturation value for large density n).

The heat flux **q** would lead to the thermoelectrical coefficient β:

$$\mathbf{q} = \int \frac{1}{2} m v^2\ \mathbf{v}\ f\, d\mathbf{v} = \int \frac{1}{2} m v^2\ \mathbf{v}\ f_1\ d\mathbf{v} = -\beta\mathbf{E}$$

Aplicación a Transporte ⑩

caso 1/ $\vec{E} = E\vec{k} = E_z\vec{k}$

con la aproximación

$$f = f_0 + f_1 + O(f_1^2) \simeq f_0 + f_1$$

despreciando términos en f_1^2 (e infinitésimos superiores) $f_1 E$, (E pequeño) quedaba

$$f_1 = \frac{q\vec{E}}{m} \cdot \frac{\partial f_0}{\partial \vec{v}}$$

$$f_0 = n(r)\left(\frac{m}{2\pi KT}\right)^{3/2} e^{-\frac{mv^2}{2KT}}$$

$$\Rightarrow \ln f_0 = \ln n(r) + \frac{3}{2}\ln\left(\frac{m}{2\pi KT}\right) - \frac{mv^2}{2KT}$$

$$\Rightarrow \frac{1}{f_0}\frac{\partial f_0}{\partial \vec{v}} = -\frac{m\vec{v}}{KT}$$

luego

$$f_1 = -\frac{-eE}{m}\frac{m}{KT}v_z$$

y $\vec{q}$ va en eje Oz :

$$q_{calor} = q_z = \int \frac{1}{2}mv^2 v_z \frac{eE}{KT} f_0(v) v_z d^3v = \beta E$$

dando

$$\beta = \int \frac{1}{2}mv^2 v_z^2 f_0(v)\, d^3v \overset{\text{esféricas}}{=}$$

$$= \frac{e}{KT}\int_{v=0}^{\infty}\int_{\theta=0}^{\pi} \frac{1}{2}mv^2\, v^2\cos^2\theta\, f_0(v)\, 2\pi v^2 \operatorname{sen}\theta\, d\theta\, dv$$

sólo en función de las integrales

$$\int_0^{\pi} \cos^2\theta \operatorname{sen}\theta\, d\theta = \frac{2}{3}$$

y de (en tablas) $\displaystyle\int_0^{\infty} v^6 e^{-\frac{v^2}{a^2}} dv = \frac{15}{16}\sqrt{\pi}\, a^2 = \frac{15}{16}\sqrt{\pi}\left(\frac{2TK}{m}\right)$

Unidad de β

$$[\beta] = \frac{[q_{cal}]}{[E]} = \frac{\left(J\cdot\frac{m}{s}\right)\cdot m^{-3}}{V/m} \rightarrow \frac{A}{m}$$

Problem (a Fokker-Planck equation as a more realistic collison term)

4) Consider the 1D kinetic equation in the convection-diffusion form:

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial v}\left[-\gamma(v-u) - \frac{\partial}{\partial v}D\right]f(v,t) \equiv -\frac{\partial j}{\partial v}$$

where

$$\int_{v=-\infty}^{\infty} f(v,t)dv = n(t), \quad n(t)u(t) = \int_{v=-\infty}^{\infty} v\, f(v,t)dv$$

Assuming that *j (a density flow)*, *f* and its derivative with respecto to *v* vanish for large $|v|$,

a) Discuss the meaning of each term (dealing *D* and γ as contants) in order to consider this equation as a collision 1-D operator (**Fokker-Planck** equation).

b) Using integration by parts on both sides of the eq. verify that n(t) =n(0) and u(t) = u(0) are constant in time and find *D* to ensure that the temperature *T* is also a constant (collisional invariant).

c) Assume that a stationary solution in the form

$$\boxed{f_s(v) = C\exp(-a(v-u)^2)}$$

exits and find it.

c) Find a simple extension of this collision term for the 3-D velocity distribution function.

$$Sol.tomar: \ \gamma = \nu_c \quad y \quad D = kT\nu_c / m$$

Problem 4) Un operador colisional convectivo-difusivo (de Fokker-Planck)

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial v}\left| -\gamma(v-u) f - D\frac{\partial f}{\partial v}\right| = -\frac{\partial j}{\partial v} \qquad (1)$$

$$= \frac{\partial}{\partial v}\left(\gamma(v-u)f\right) + D\frac{\partial^2 f}{\partial v^2}$$

a) $-\gamma(v-u)$ representa una aceleración de frenado viscosa, cada partícula sufre fuerza de frenado proporcional a su velocidad (relativa a la velocidad media u) $a = -\gamma(v-u) = -\gamma w$

$D\frac{\partial^2 f}{\partial v^2}$ efecto difusivo (en velocidades)

por efecto de colisiones, por sí solo harían tender a f a cero cont.

b) Para ver $n(t)$, integrando miembro a miembro (1):

$$\int_{-\infty}^{\infty}\frac{\partial f}{\partial t}dv = -\int_{-\infty}^{\infty}\frac{\partial j}{\partial v}dv \Rightarrow \frac{\partial n}{\partial t} = -\left(j\right)_{-\infty}^{\infty} = 0 \Rightarrow \underline{\underline{n=n_0}}$$

constante

Para el momento, multiplicando por vm e integrando

$$\int_{-\infty}^{\infty}\frac{\partial}{\partial t}mvf\,dv = -\int_{-\infty}^{\infty} mv\underbrace{\frac{\partial j}{\partial v}}dv \Rightarrow$$

$$m\frac{\partial}{\partial t}nu = mn_0\frac{\partial u}{\partial t} = -\left\{mvj\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} j\,dv\right\} =$$

$$= 0 + \int_{-\infty}^{\infty}\left(-\gamma(v-u)f - D\frac{\partial f}{\partial v}\right)dv = -\gamma n_0(u-u) - Df\Big|_{-\infty}^{\infty}$$

$$= 0 \Rightarrow u = u_0$$

constante

Para ver la evolución de la energía, se opera igual, multiplicando por $\frac{1}{2}mv^2$

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\frac{1}{2}m f\, dv = \frac{\partial}{\partial t}\left(\frac{1}{2}kT\right) = -\int_{-\infty}^{\infty}\frac{1}{2}mv^2\frac{\partial j}{\partial v}\, dv$$

$\Rightarrow$ operando:

$$\frac{\partial}{\partial t}\frac{1}{2}kT = 2k\left[D - 2T\gamma + \gamma \overset{0}{u_0^2}\right]$$

Puede elegirse D para que $\frac{dT}{dt} = 0$.
y si $u(0) = u_0$, se conservan la densidad, el momento y la energía (si sólo actúa para la variación de f este término colisional).

c/ Si la solución a ensayar es

$$f_s = C\exp\{-a(v-u)^2\}$$

$$j = -\gamma(v-u)f_s - D\frac{\partial f_s}{\partial v} = f_s\{\gamma(v-u) - 2D(v-u)a\} = 0$$

basta tomar

$$a = \frac{\gamma}{2D}$$

Además: $n_0 = \int_{-\infty}^{\infty} f_s\, dv$ etc...

da, identificando, $f_s = n_0\sqrt{\frac{m}{2\pi kT}}\, e^{-\frac{mv^2}{2kT}}$

$$\boxed{D = \gamma\frac{kT}{m}}$$ (coeficiente de difusión de Einstein).

d/ En 3-D basta tomar $\mathbb{D} = D\mathbb{1}$ y fricción γ isótropa:

$$C(f) = -\frac{\partial}{\partial\vec{v}}\cdot\left\{-\gamma(\vec{v}-\vec{u})f - D\frac{\partial f}{\partial\vec{v}}\right\} = +\gamma\frac{\partial}{\partial\vec{v}}\cdot(\vec{v}-\vec{u})f + D\nabla_v^2 f$$

es lo más simple (Ecuación del calor, otra vez)

Nota 1: Solución a la ecuación de Boltzmann.

N ①

Para la ecuación

$$\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial v} = -\nu\left(f - f_0(v)\right)$$

existe solución (al menos formal) para dar $f(v,t)$ conocida $f(v,t=0) = g(v)$ y la función de equilibrio referente $f_0(v)$, una maxwelliana, por ejemplo.

El procedimiento de encontrar las "curvas características" de la ecuación asociadas a f da las identidades:

$$\underbrace{dt = \frac{dv}{a}}_{①} \underbrace{= -\frac{1}{\nu}\left(f - f_0(v)\right)}_{②}$$

suponiendo el campo de aceleraciones a constante:

de ① $\quad v - at = v_0 \quad$ (primera constante de integración)

de ② $\quad \dfrac{df}{dv} = -\dfrac{\nu}{a}\left(f - f_0\right) = -\dfrac{\nu}{a} f + \dfrac{\nu}{a} f_0(v) \quad$ (lineal)

con solución:

$$f = K e^{-\nu v/a} + \frac{\nu}{a} e^{-\frac{\nu v}{a}} \int_0^v f_0(u)\, e^{\frac{\nu u}{a}}\, du$$

donde K es función de v_0, $K = K(v_0) = K(v-at)$

despejando K de $f(v,0) = g(v)$:

$$f(v,t) = g(\cancel{v_0}) = g(v-at)\, e^{-\nu t} + \frac{\nu}{a}\int_{v-at}^{v} e^{-\frac{\nu}{a}(v-u)} f_0(u)\, du$$

que cumple la ecuación (verificar por sustitución)

Para la solución independiente del tiempo hay que resolver la misma ecuación con $\frac{\partial f}{\partial t} = 0$

lo que da:

$$f_{esta.} = \frac{\nu}{a} \int_{v_0}^{v} e^{-\frac{\nu}{a}(v - v')} f_0(v')\, dv'$$

a la que se llega también haciendo $t \to \infty$ en la solución a $f(v, t)$ con la suposición de que $|g| \to 0$ si su argumento $|v - at| \to \infty$

Un caso particular:

$$a = \frac{qE}{m}$$

y

$$f_0 = \frac{n_0}{\sqrt{KT\pi/m}} e^{-\frac{mv^2}{2KT}}$$

Nota 2:

La ecuación de Fokker-Planck unidimensional

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial v}\left\{ -\nu(v-u) f - \frac{\partial}{\partial v} D f \right\} \qquad (1)$$

admite solución exacta para D y u constantes (o funciones de t) dada la solución general para condición inicial $f(v,0) = n\,\delta(v-v_0)$ se puede tener la solución para cualquier $f(v,0)$ por

$$f(v,t) = \int_{-\infty}^{\infty} f(v',0)\, G(v|v';t)\, dv'$$

donde G es la Función de Green del problema (1) que satisface la misma ecuación con condición $G(v|v';t=0) = \delta(v-v')$, y es una Gaussiana de media $m(t)$ y varianza $\sigma^2(t)$

$$G(v|v';t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(v-m)^2}{2\sigma^2} \right\}$$

donde $m = v' e^{-\nu t} \ldots$ y $\sigma^2(t)$ pueden encontrarse por sustitución en ①. (Tedioso).

Para cualquier $f(v,0)$, hay solución estacionaria independiente de $f(v,0)$ si $|f(v)| \to 0$ con $|v| \to \infty$, solución de la ecuación lineal:

$$-\nu(v-u) f = D \frac{\partial f}{\partial v} \rightarrow \log(f) = -\frac{\nu(v-u)^2}{2D} + cte$$

$$f = C_0 \exp\left\{ -\frac{\nu(v-u)^2}{2D} \right\}$$